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Accurate Solutions of the Navier–Stokes Equations Despite Unknown Outflow Boundary Data

JAN NORDSTRÖM

FFA, The Aeronautical Research Institute of Sweden, P.O. Box 11021, S-161 11 Bromma, Sweden*

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A very common procedure when constructing boundary conditions for the time-dependent Navier–Stokes equations at artificial boundaries is to extrapolate the solution from grid points near the boundary to the boundary itself. For supersonic outflow, where all the characteristic variables leave the computational domain, this leads to accurate results. In the case of subsonic outflow, where one characteristic variable enters the computational domain, one cannot in general expect accurate solutions by this procedure. The problem with outflow boundary operators of extrapolation type at artificial boundaries with errors in the boundary data of order one will be investigated. Both the problem when the artificial outflow boundary is located in essentially uniform flow and the situation when the artificial outflow boundary is located in a flow field with large gradients are discussed. It will be shown, that in the special case when there are large gradients tangential to the boundary, extrapolation methods can be used even in the subsonic case. © 1995 Academic Press, Inc.

1. INTRODUCTION

In many computational problems one is faced with infinite domains, which for computational reasons must be made finite. One possibility is to map the infinite domain onto a finite one (see, for example, Grosch and Orzag [1]). Solution methods based on this technique are not considered here. Another possibility, is to introduce an artificial boundary Γ in order to reduce the infinite computational domain Ω_∞ to a finite one Ω. The introduction of the artificial boundary makes it necessary to formulate appropriate artificial boundary conditions.

Artificial boundary conditions are used in many fields of computational physics such as gas dynamics, hydrodynamics, meteorology, elasticity, acoustics, electromagnetism, etc.. The different practical applications require different features of the artificial boundary conditions, but there are some common points.

One has to assume that the original problem in the unbounded domain Ω_∞ is well posed. The introduction of the artificial boundary Γ and the corresponding boundary conditions must also be such that the new problem in the bounded domain Ω is well posed; see Kreiss [2] for a discussion on well-posedness.

The solutions of the two problems should differ slightly from each other in Ω. Let v be the solution in the bounded domain Ω and u the solution of the original problem in Ω_∞. If the problem in the bounded domain Ω is well posed and the solution of the original problem approximately satisfies the artificial boundary condition on Γ, then the difference $w = u - v$ is small (see Hagstrom [3], Bayliss and Turkel [4], and Halpern [5]). In this paper it is shown that accurate solutions v can be obtained even if the solution u is poorly approximated by the artificial boundary conditions. In many cases a simplified way of estimating the difference between the two solutions are used. For example, in the case of the absorbing boundary conditions discussed below, the difference between the two solutions in the bounded and unbounded domain is considered small if the reflection of the outgoing waves are small.

The problem with artificial boundaries will be solved numerically and hence it is important that stable discrete boundary conditions can be constructed from the continuous ones. It is of course possible to develop artificial boundary conditions directly for the discrete problem; see, for example, Higdon [6]. Artificial boundary conditions have also been formulated for nonlinear problems (see Hedstrom [9], Thompson [10, 11], Hagstrom [12], and Dutt [13]). Most researcher's, however, have developed their boundary conditions for the continuous problem in the linear regime and discretised later.

Many methods for constructing artificial boundary conditions lead to exact conditions in the form of integral relations. The boundary conditions are normally non-local in both time and space and, hence, not useful in practical calculations. The localisation of the boundary conditions are made by approximating the integral relation at certain dominating frequencies. A particular high frequency limit was considered by Engquist and Majda [7, 8] while Hagstrom [14, 15] identified the dominant frequencies in dissipative systems by the method of steepest decent.

It is important to note that information from the solution in the exterior domain $\Omega_\infty - \Omega$ to the solution in the domain Ω is transferred only via the artificial boundary conditions. This

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means that some a priori knowledge of the exterior solution is necessary. The type of knowledge about the exterior solution and how to transfer this knowledge to the artificial boundary conditions is, roughly speaking, what separates the different types of artificial boundary conditions in the literature. For a more detailed discussion on these matters see Gustafsson and Kreiss [16].

If accurate data are available through measurements, or if the computational domain is so large that data obtained from the state at infinity are sufficiently accurate, then one can use well-posed boundary conditions of standard type. Gustafsson and Sundström [17], Oliger and Sundström [18], Engquist and Gustafsson [19], and Nordström [20, 21] all used the energy method to show well posedness of the Euler or Navier–Stoke's equations with constant coefficients. The boundary conditions were chosen so that an energy estimate of the dissipative type was obtained.

In most cases, however, accurate data are not available. The principle of no reflection, i.e., where only outgoing waves from the inner domain are allowed, leads to the so-called absorbing boundary conditions first developed by Engquist and Majda [7, 8]. They are constructed in such a way that outgoing disturbances in the form of waves are not reflected back into the computational domain. Note that if ingoing waves are present, i.e., signals are entering the computational domain, then the absorbing boundary conditions require accurate boundary data. Gustafsson [22] generalized the absorbing boundary conditions to the case with non-zero data outside the computational domain.

Another way to transfer information about the exterior solution to the artificial boundary conditions is to use a simplified form of the equations in the exterior domain and formally solve the equations exactly. The form of the solution in the exterior domain is then used to construct boundary conditions for the true equations in the inner domain. Such procedures usually lead to non-local conditions (see, for example, Ferm and Gustafsson [23], Ferm [24, 25], Keller and Givioli [26], and Hagstrom [27]).

In many cases it is impossible to use methods based on simplified models outside Γ. For example, the geometry may be non-trivial, not permitting any simplifications, or the flow field might have large gradients. The question then is whether or not one can construct boundary operators L giving accurate solutions v in Ω despite the fact that the true data g in $Lv = g$ are not known.

A very common procedure is to extrapolate the solution from grid points near the boundary to the boundary itself. Linear extrapolation, say, is an approximation of the condition $\partial^2 v/\partial n^2 = 0$, where n is normal to the boundary. For first- and second-order systems, like the Euler and Navier–Stokes equations, one cannot expect accurate solutions if ingoing characteristics are present. Kreiss [28] has investigated these matters further. For most first-order derivatives in the boundary conditions, where an energy estimate holds, conditions are given such that the boundary layer solutions are suppressed.

In [29] the steady Navier–Stokes equations were analysed and it was shown that extrapolation methods can be used with good results also in the subsonic case if there are large gradients tangential to the boundary. A time-dependent model problem for the Navier–Stokes equations was analysed in [30, 31]. Based on that analysis and numerical experiments it was proposed that extrapolation methods can be used also for the time-dependent Navier–Stokes equations. In this paper the theoretical basis for the result proposed in [30, 31] is given.

The remainder of this paper will proceed as follows. In Section 2.1 the general problem with artificial boundaries and inaccurate boundary data is discussed. A linear model problem is derived Section 2.2 and the main result of the paper is given. In Section 2.3 the problem with derivative boundary conditions at artificial boundaries in uniform flow is considered. The influence of large gradients is included in the analysis in Section 2.4. Numerical experiments that exemplify the results obtained in Section 2.3 are given in Section 3.1. The results of Section 2.4 are compared with the results from numerical experiments in Section 3.2. Finally, in Section 4 we sum up and draw conclusions.

2. ANALYSIS

2.1. *Artificial Boundaries and Inaccurate Boundary Data*

We wish to solve the initial boundary value problem for the flow around a solid body in the unbounded domain Ω_∞ (see Fig. 1)

$$\begin{aligned} u_t &= P(u)u + F(x,t), && x \in \Omega_\infty, t \geq 0, \\ u &= f(x), && x \in \Omega_\infty, t = 0, \\ L_S u &= g_S(t), && x \in \Gamma_S, t \geq 0, \end{aligned} \tag{1}$$

where P is the differential operator and L_S is the boundary operator. F, f, g_S are the forcing function, the initial function, and the boundary data on the solid boundary, respectively. The

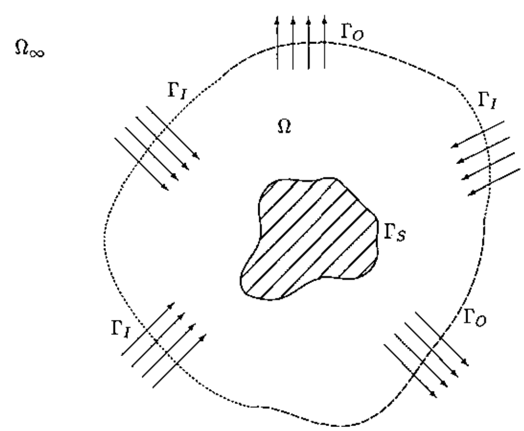

FIG. 1. Geometry definition for the flow around a solid body.

quantities F, f, g_S are considered as the data of the problem.

In practice we introduce artificial boundaries and try to solve the problem in the bounded domain Ω,

$$\begin{aligned} v_t &= P(v)v + F(x,t), && x\in\Omega, t\ge 0,\\ v &= f(x), && x\in\Omega, t=0,\\ L_S v &= g_S(t), && x\in\Gamma_S, t\ge 0,\\ L_O v &= g_O(t), && x\in\Gamma_O, t\ge 0,\\ L_I v &= g_I(t), && x\in\Gamma_I, t\ge 0, \end{aligned} \tag{2}$$

where the subscripts I, O, S denote an inflow boundary, an outflow boundary, and a solid boundary, respectively. For smooth flow, the difference $w = u - v$ is an approximation of the solution to the linearised problem,

$$\begin{aligned} w_t &= P(u)w, && x\in\Omega, t\ge 0,\\ w &= 0, && x\in\Omega, t=0,\\ L_S w &= 0, && x\in\Gamma_S, t\ge 0,\\ L_O w &= \tilde g_O(t), && x\in\Gamma_O, t\ge 0,\\ L_I w &= \tilde g_I(t), && x\in\Gamma_I, t\ge 0. \end{aligned} \tag{3}$$

Note that the only data left in (3) is the boundary data. Compatibility at $t = 0$ requires $\tilde g_O(0) = \tilde g_I(0) = 0$.

The formulation (3) will be used to analyse boundary conditions on Γ_O close to Γ_S although, strictly speaking, the region of smooth flow is located some distance away from Γ_S. Normally one knows boundary data with good accuracy at inflow boundaries, while knowledge of data at outflow boundaries often are lacking. In other words one often has $\tilde g_I = g_I - L_I u \approx 0$ and $\tilde g_O = g_O - L_O u = O(1)$.

In this paper outflow boundary operators L_O of extrapolation (or derivative) type with errors in the boundary data of order one will be investigated. Both the problem when the artificial outflow boundary is located far away from the solid body in essentially uniform flow and the situation when the artificial outflow boundary intersects the solid body in a flow field with large gradients (see Fig. 2) will be discussed.

2.2. *The Linearised Navier–Stokes Equations*

The Navier–Stokes equations in non-dimensional form are

$$\begin{aligned} \rho_t &= -u\rho_x - v\rho_y - \rho(u_x+v_y)\\ u_t &= -uu_x - vu_y - p_x/\rho + (\varepsilon/\rho)[(\theta u_x+\lambda v_y)_x + (\mu u_y+\mu v_x)_y]\\ v_t &= -uv_x - vv_y - p_y/\rho + (\varepsilon/\rho)[(\mu u_y+\mu v_x)_x + (\theta v_y+\lambda u_x)_y]\\ T_t &= -uT_x - vT_y - (\gamma-1)T(u_x+v_y)\\ &\quad + (\varepsilon/\rho)[(\varphi T_x)_x + (\varphi T_y)_y + M_\infty^2\Phi]\\ \Phi &= \gamma(\gamma-1)[u_x(\theta u_x+\lambda v_y) + \mu(u_y+v_x)^2 + v_y(\lambda u_x+\theta v_y)], \end{aligned} \tag{4}$$

where $p = \rho T/(\gamma M_\infty^2)$, $c^2 = T/M_\infty^2$, $\theta = \lambda + 2\mu$, and $\varphi = \gamma\kappa/\mathrm{Pr}$.

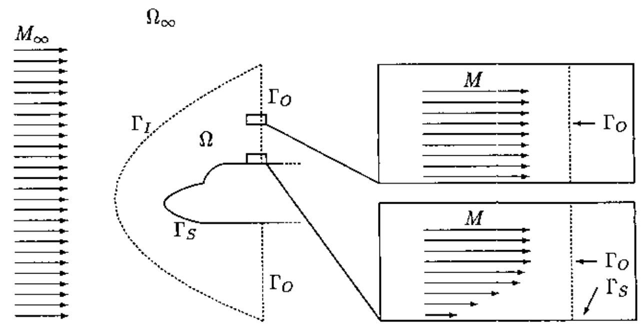

FIG. 2. Geometry definition for the flow around the nose region of a vehicle.

The dependent variables and parameters ρ, u, v, T, p, c, M_∞, μ, λ, κ, Pr, γ, and ε are respectively the density, x and y components of the velocity, the temperature, the pressure, the speed of sound, the free stream Mach number, the shear and second viscosity, the coefficient of heat conduction, the Prandtl number, the ratio of specific heats, and the inverse Reynolds number.

Let us consider the flow field schematically depicted in Fig. 2 and focus on the intersection between the artificial boundary Γ_O and the solid boundary Γ_S. With a no-slip condition at Γ_S the flow normal to Γ_O is subsonic sufficiently close to Γ_S even if $M \gg 1$. To study the effect of extrapolation boundary conditions on Γ_O the Navier–Stokes equations are linearised in a domain where large gradients are present in the basic flow field.

Let x be the coordinate along the solid wall and y the coordinate normal to the wall. In the analysis below a time-dependent disturbance around an almost steady flow $\bar\phi = (\bar\rho, \bar u, \bar v, \bar T)^T$ with small streamwise x-gradients and large transversal y-gradients is investigated. The equation for the perturbation $w = (\rho, u, v, T)^T$ can formally be written

$$w_t + \bar A_{10}w_x + \bar A_{01}w_y + \bar B_{00}w = \varepsilon\{\bar A_{20}w_{xx} + \bar A_{11}w_{xy} + \bar A_{02}w_{yy}\}, \tag{5}$$

where the matrices $\bar A_{ij}$ are the ones normally included in a linearised equation. In our case we have also kept the zero order terms $\bar B_{00}w$, where

$$\bar B_{00} = \begin{pmatrix} \bar u_x + \bar v_y & \bar\rho_x & \bar\rho_y & 0\\ -(\bar c^2/(\gamma\bar\rho))(\bar\rho_x/\bar\rho) & \bar u_x & \bar u_y & (1/\gamma M_\infty^2)(\bar\rho_x/\bar\rho)\\ -(\bar c^2/(\gamma\bar\rho))(\bar\rho_y/\bar\rho) & \bar v_x & \bar v_y & (1/\gamma M_\infty^2)(\bar\rho_y/\bar\rho)\\ 0 & \bar T_x & \bar T_y & (\gamma-1)(\bar u_x+\bar v_y) \end{pmatrix}. \tag{6}$$

Only the dominating components of $\bar B_{00}$ will be included in the following analysis.

From boundary layer theory (see White [32]), it is known that $\bar\rho_y$, $\bar u_y$, $\bar T_y$ are large compared with $\bar\rho_x, \bar u_x, \bar v_x$, $\bar T_x$, $\bar p_x$, $\bar v_y$, $\bar p_y$ at sufficiently high free stream Mach numbers. Furthermore, the magnitude of $\overline{\rho}_y, \overline{u}_y, \overline{T}_y$ are increasing with increasing Reynolds number (or decreasing ε) and, hence, the following assumption is made.

Assumption 2.1. The first-order gradients in the basic flow field are assumed to satisfy

$$\overline{\rho}_x, \overline{u}_x, \overline{v}_x, \overline{T}_x, \overline{p}_x = 0, \quad \overline{v}_y, \overline{p}_y = 0, \quad |\overline{\rho}_y|, |\overline{u}_y|, |\overline{T}_y| \geq C_1 \varepsilon^{-q}, \quad (7)$$

where C_1 is a bounded non-zero constant and $0 < q < 1$.

The matrix $\overline{B}_{00}$ multiplying the zero order terms in the following analysis will be $\varepsilon^{-q}\,\overline{B}$, where

$$\overline{B} = \begin{pmatrix} 0 & 0 & \tilde{\rho}_y & 0 \\ 0 & 0 & \tilde{u}_y & 0 \\ -(\overline{c}^2/(\gamma\overline{\rho}))(\tilde{\rho}_y/\overline{\rho}) & 0 & 0 & (1/(\gamma M_\infty^2))(\tilde{\rho}_y/\overline{\rho}) \\ 0 & 0 & -M_\infty^2\overline{c}^2(\tilde{\rho}_y/\overline{\rho}) & 0 \end{pmatrix}. \quad (8)$$

The components of $\overline{B}$ are now all of order one. The relation $\tilde{p}_y = 0$ has been used to replace $\tilde{T}_y$ by $-M_\infty^2\overline{c}^2(\tilde{\rho}_y/\overline{\rho})$.

Consider the Cauchy problem for Eq. (5) without zero order terms. Fourier transformation in y and the use of the energy-method (see Nordström [21]) leads to

$$\|\hat{w}\|^2 + 2\varepsilon\alpha \int_0^T \{\|\hat{w}_x^{(2)}\|^2 + |\omega|^2\|\hat{w}^{(2)}\|^2\}\,dt \leq \|\tilde{f}\|^2,$$

where $w^{(2)} = (u, v, T)^T$, f is the initial data, ω is the dual variable to y, and α is a constant of order one. The error estimate for the Cauchy problem is less sharp for perturbations with small variations in y (corresponding to $\hat{w}(\omega \to 0)$). A similar result can be expected also for the corresponding initial boundary value problem including zero order terms. For that reason and in order to make the analysis more straightforward we will restrict ourselves to perturbations independent of y. The numerical results shown later confirm the relevance of this restriction.

We obtain (see Eq. (5)), the linearised unsteady one-dimensional equation for a quarter space

$$\begin{aligned} w_t &= (\overline{P}_1 + \overline{P}_0)w, && x \leq 0, t \geq 0, \\ w &= 0, && x \leq 0, t = 0, \\ \partial^r w^{(2)}/\partial x^r &= g(t), && x = 0, t \geq 0, \\ w &\to 0, && x \to -\infty, t \geq 0, \end{aligned} \quad (9)$$

where $\overline{P}_1(\partial/\partial x) = -\overline{A}_{10}\partial/\partial x + \overline{A}_{20}\partial^2/\partial x^2 = -\overline{A}\partial/\partial x + \varepsilon\overline{C}\partial^2/\partial x^2$, $\overline{P}_0 = -\overline{B}_{00} = -\varepsilon^{-q}\,\overline{B}$, $w = (\rho, u, v, T)^T$, $w^{(2)} = (u, v, T)^2$, $g = (g_1, g_2, g_3)^T$, $r \geq 0$, $\overline{u} > 0$, $0 < \varepsilon \ll 1$, and

$$\overline{A} = \begin{pmatrix} \overline{u} & \overline{\rho} & 0 & 0 \\ \overline{c}^2/\gamma\overline{\rho} & \overline{u} & 0 & 1/(\gamma M_\infty^2) \\ 0 & 0 & \overline{u} & 0 \\ 0 & (\gamma-1)\overline{c}^2 M_\infty^2 & 0 & \overline{u} \end{pmatrix},$$
$$\overline{C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \overline{\theta}/\overline{\rho} & 0 & 0 \\ 0 & 0 & \overline{\mu}/\overline{\rho} & 0 \\ 0 & 0 & 0 & \overline{\varphi}/\overline{\rho} \end{pmatrix}. \quad (10)$$

The matrices $\overline{A}, \overline{B}, \overline{C}$ given by (8), (10) are now of order one. There is subsonic outflow at $x = 0$ if $\overline{u} < \overline{c}$ and $\overline{c} > \overline{u}$ indicates supersonic outflow. Compatibility at $t = 0$ requires $g(0) = 0$. The problem (9) with $\overline{P}_0 \neq 0$ is called the disturbed problem. If $\overline{P}_0$ is 0, (9) is referred to as the undisturbed problem.

Recall that w is the difference between the solution u in the unbounded domain and the solution v in the bounded domain with the artificial boundary at $x = 0$. The problem (9) corresponds to (3) above.

Strikwerda [33] has shown that the two-dimensional Navier–Stokes equations require three boundary conditions at an outflow boundary while one normally extrapolates all four variables in a numerical calculation. The Navier–Stokes equations consist of a hyperbolic equation for $w^{(1)} = \rho$ coupled to a set of parabolic equations for $w^{(2)} = (u, v, T)^T$. It can be shown that well-posedness of the decoupled hyperbolic and parabolic equations leads to well-posedness of the whole system including the coupling terms (see Kreiss and Lorenz [34]). The decoupled parabolic equations require three boundary conditions involving $w^{(2)}$, while no boundary condition is necessary for the hyperbolic equation in the outflow case and, hence, the three boundary conditions in (9) are imposed on $w^{(2)}$.

The Laplace-transformed version of (9) is

$$\begin{aligned} (SI - \varepsilon(\overline{P}_1 + \overline{P}_0))\hat{w} &= 0, && x \leq 0 \\ \partial^r\hat{w}^{(2)}/\partial x^r &= \hat{g}, && x = 0 \\ \hat{w} &\to 0, && x \to -\infty, \end{aligned} \quad (11)$$

where the notation $S = \eta + i\xi = \varepsilon s = \varepsilon(\tilde{\eta} + i\tilde{\xi})$ is introduced. The problem (11) is a system of ordinary differential equations and the solution can formally be written, $\hat{w} = \sum_j Q_j(x)\exp(\kappa_j x/\varepsilon)$, where κ is a root of the characteristic equation

$$\mathrm{Det}\{SI - \varepsilon(\overline{P}_0 + \overline{P}_1(\kappa/\varepsilon))\} = 0. \quad (12)$$

In the general case Q_j is a polynomial in x with vector coefficients of degree $n - k$, where n is the algebraic multiplicity of κ_j and k is the number of linearly independent eigenvectors (the geometric multiplicity) associated with κ_j. To satisfy the condition that $\hat{w} \to 0$ when $x \to -\infty$ only κ with $\mathrm{Re}(\kappa) > 0$ can be accepted. If necessary the superscript + for eigenvalues with positive real part are used.

For the two-dimensional Navier–Stokes equations at an outflow boundary there are three κ^+ (regardless of subsonic or supersonic flow) for $\mathrm{Re}(S)$ sufficiently large; see [33]. Thus, the solution has three unknown coefficients $\sigma(S) = (\sigma_1, \sigma_2, \sigma_3)^T$ to be determined by the boundary conditions. This leads to a linear system of equations

$$E(S)\sigma(S) = \hat{g}. \tag{13}$$

A major part of this paper is devoted to the analysis of (13). For later reference the notations

$$\kappa^R_{\min} = \kappa(S)^R_{\min} = \min_j \mathrm{Re}(\kappa_j^+), \quad j = 1, 2, 3, \tag{14}$$

$$\kappa_{\min} = \kappa(S)_{\min} = \min_j |\kappa_j^+|, \quad j = 1, 2, 3,$$

are introduced.

The basic results of this paper are the following.

THEOREM 2.1. *The solution of* (9) *with* $\overline{P}_0 \neq 0$ *satisfies*

$$\int_0^T |w(x, t)|^2 \exp(-2\eta t)\,dt \le c_1 \varepsilon^{2r} \exp\left(\frac{\alpha_1 x}{\varepsilon}\right) \int_0^T |g(t)|^2 \exp(-2\eta t)\,dt \tag{15}$$

if $\overline{u} > \overline{c}$ *and*

$$\int_0^T |w(x, t)|^2 \exp(-2\eta t)\,dt \le c_2 \varepsilon^{2rq} \exp((\alpha_2 \eta + \alpha_3 \varepsilon^{(1-2q)})x) \int_0^T |g(t)|^2 \exp(-2\eta t)\,dt \tag{16}$$

if $\overline{u} < \overline{c}$. *Here* $\eta \ge 0$ *and* $x \le 0$. *The constants* $c_1, c_2, \alpha_1, \alpha_2, \alpha_3$ *are positive and independent of* η *and* ε.

THEOREM 2.2. *The solution of* (9) *with* $\overline{P}_0 = 0$ *and* $\overline{u} > \overline{c}$ *satisfies*

$$\int_0^T |w(x, t)|^2 \exp(-2\eta t)\,dt \le c_1 \varepsilon^{2r} \exp\left(\frac{\alpha_1 x}{\varepsilon}\right) \int_0^T |g(t)|^2 \exp(-2\eta t)\,dt \tag{17}$$

where $\eta \ge 0$ *and* $x \le 0$. *The solution of* (9) *with* $\overline{P}_0 = 0$ *and* $\overline{u} < \overline{c}$ *satisfies*

$$\int_0^T |w(x, t)|^2 \exp(-2\eta t)\,dt \le c_2 (1/\eta)^{2r} \exp(\alpha_2 \eta x) \int_0^T |g(t)|^2 \exp(-2\eta t)\,dt \tag{18}$$

and

$$\int_0^\infty |w(x, t)|^2 \exp(-2\eta t)\,dt \ge c_3 \left(\frac{|\overline{u} - \overline{c}|^2}{\eta^2 + c_4}\right)^r \exp\left(\frac{2\eta x}{|\overline{u} - \overline{c}|}\right), \tag{19}$$

where $\eta \ge \eta_0 > 0$ *and* $x \le 0$. *The constants* $c_1, c_2, c_3, c_4, \alpha_1, \alpha_2$ *are positive and independent of* η *and* ε.

Theorem 2.1 means that:

- The error w in the solution is bounded by the boundary data.
- If $r > 0$, then the error is very small in both the supersonic and subsonic case.

Theorem 2.2 means that:

- The error w in the solution is bounded by the boundary data.
- If $r > 0$, then the error is very small in the supersonic case.
- The error is not small in the subsonic case.

The type of problem (see (3), (9)) that is investigated in this paper has a correct solution at $t = 0$ (i.e., the initial function is zero). The only source of error, the boundary data, is not likely to cause large errors in the solution for very short times. However, after an initial short time period that might happen and, hence, estimates of the type given in Theorems 2.1 and 2.2 are useful only if they lead to small errors for long times. A large η in the estimates above means that one examines the solution for short times while a small η means that one focuses on the long time behavior of the solution.

Note that the estimate (19) leads to errors of order one also for $r > 0$. This corresponds to the well-known fact that extrapolation of all variables at a subsonic outflow boundary normally leads to inaccurate solutions. Note also that the factors $\exp(\alpha\eta x)$, $\alpha > 0$, in the estimates (18), (19) does not mean that w itself has a boundary layer at $x = 0$.

2.3. The Undisturbed Problem without Zero Order Terms

Consider (12) with $\overline{P}_0 = 0$, $\mathrm{Det}\{SI - \varepsilon \overline{P}_1(\kappa/\varepsilon)\} = f_0(\kappa) = f_{1,3}(\kappa) \times f_2(\kappa)$, where

$$\begin{aligned} f_{1,3}(\kappa) = {} & \kappa^5\{\overline{u}\overline{\theta}\overline{\varphi}/\overline{\rho}^2\} + \kappa^4\{S\overline{\theta}\overline{\varphi}/\overline{\rho}^2 - \overline{u}^2(\overline{\theta} + \overline{\varphi})/\overline{\rho} + \overline{c}^2\overline{\varphi}/\gamma\overline{\rho}\} \\ & + \kappa^3\{\overline{u}(\overline{u}^2 - \overline{c}^2) - 2\overline{u}S(\overline{\theta} + \overline{\varphi})/\overline{\rho}\} \\ & + \kappa^2\{(3\overline{u}^2 - \overline{c}^2)S - S^2(\overline{\theta} + \overline{\varphi})/\overline{\rho}\} + \kappa\{3\overline{u}S^2\} + S^3 \end{aligned} \tag{20}$$

$$f_2(\kappa) = \kappa^2\{-\overline{\mu}/\overline{\rho}\} + \kappa\{\overline{u}\} + S \tag{21}$$

$$\begin{aligned} f_0(\kappa) = {} & \kappa^7\{a_{70}\} + \kappa^6\{a_{60} + a_{61} S\} + \kappa^5\{a_{50} + a_{51} S\} \\ & + \kappa^4\{a_{40} + a_{41} S + a_{42} S^2\} + \kappa^3\{a_{31} S + a_{32} S^2\} \\ & + \kappa^2\{a_{22} S^2 + a_{23} S^3\} + \kappa^1\{a_{13} S^3\} + a_{04} S^4 \end{aligned} \tag{22}$$

and

$$a_{70} = -\overline{u}\overline{\theta}\overline{\mu}\overline{\varphi}/\overline{\rho}^3, \qquad a_{60} = (\overline{u}^2(\overline{\mu}\overline{\theta} + \overline{\theta}\overline{\varphi} + \overline{\varphi}\overline{\mu}) - \overline{c}^2\overline{\mu}\overline{\varphi}/\gamma)/\overline{\rho}^2$$

$$a_{61} = -\overline{\mu}\overline{\theta}\overline{\varphi}/\overline{\rho}^3, \qquad a_{50} = (\overline{u}\overline{c}^2(\overline{\mu} + \overline{\varphi}/\gamma) - \overline{u}^3(\overline{\mu} + \overline{\theta} + \overline{\varphi}))/\overline{\rho}$$

$$a_{51} = 2\overline{u}(\overline{\mu}\overline{\theta} + \overline{\theta}\overline{\varphi} + \overline{\mu}\overline{\varphi})/\overline{\rho}^2 \qquad a_{40} = \overline{u}^2(\overline{u}^2 - \overline{c}^2)$$

$$a_{41} = (\overline{c}^2(\overline{\mu} + \overline{\varphi}/\gamma) - 3\overline{u}^2(\overline{\mu} + \overline{\theta} + \overline{\varphi}))/\overline{\rho}, \quad a_{42} = (\overline{\mu}\overline{\theta} + \overline{\theta}\overline{\varphi} + \overline{\varphi}\overline{\mu})/\overline{\rho}^2$$

$$a_{31} = 4\overline{u}^3 - 2\overline{u}\overline{c}^2, \qquad a_{32} = -3\overline{u}(\overline{\mu} + \overline{\theta} + \overline{\varphi})/\overline{\rho}$$

$$a_{22} = 6\overline{u}^2 - \overline{c}^2, \qquad a_{23} = -(\overline{\mu} + \overline{\theta} + \overline{\varphi})/\overline{\rho}$$

$$a_{13} = 4\overline{u}, \qquad a_{04} = 1.$$

If $\mathrm{Re}(S) > 0, f_2(\kappa) = 0$ has one root κ_2^+ with positive real part. The fifth-order equation $f_{1,3}(\kappa) = 0$ gives us two roots, κ_1^+, κ_3^+. A simple algebraic check reveals that $f_{1,3}(\kappa_2^+) \neq 0$, $f_2(\kappa_j^+) \neq 0, j = 1, 3$, for all S with $\mathrm{Re}(S) > 0$ and consequently no triple roots with $\mathrm{Re}(\kappa) > 0$ exist. If necessary the subscript s and d will be used to indicate quantities related to the single and double root solutions, respectively.

The existence of a double root κ^+ is possible and the following lemma says where in the complex S-plane it might be found.

LEMMA 2.1. *Consider* (22) *with* $\mathrm{Re}(S) > 0$. *If a double root* $\kappa^+ = \kappa^+(S^*)$ *exists then*

$$\delta_0 < |S^*| < \delta_1, \quad \delta_2 < |\kappa^+| < \delta_3, \tag{23}$$

where δ_0, δ_1, δ_2, δ_3 *are positive constants.*

The proof is given in the Appendix. Some additional information about the eigenvalues is necessary.

LEMMA 2.2. *Let* $S = \eta + i\xi$, $\eta \geq 0$, $|S| \geq \delta_0$. *The eigenvalues* $\kappa^+ = \alpha + i\beta$ *obtained from* $f_0(\kappa) = f_{1,3}(\kappa) \times f_2(\kappa) = 0$ *satisfy*

$$\begin{aligned} &\alpha \geq C_1\delta_0^2, \quad |\beta|/\alpha \leq C_2/\delta_0, \quad \delta_0 > 0, \overline{u} \leq \overline{c}, \\ &\alpha \geq C_3, \quad |\beta|/\alpha \leq C_4, \quad \delta_0 = 0, \overline{u} > \overline{c}, \end{aligned} \tag{24}$$

where the constants C_1, C_2, C_3, C_4 *are positive and independent of* S. *The functions* $f_{1,3}$ *and* f_2 *are given by* (20) *and* (21).

LEMMA 2.3. *Let* $S = \eta + i\xi$, $\eta \geq \eta_0$. *The eigenvalues* $\kappa^+ = \alpha + i\beta$ *obtained from* $f_0(\kappa) = f_{1,3}(\kappa) \times f_2(\kappa) = 0$ *satisfies*

$$\begin{aligned} &\kappa_{\min}^R \geq C_1\eta_0, \quad \kappa_{\min} \geq C_2\eta_0, \quad |\beta|/\alpha \leq C_3, \quad \eta_0 > 0, \quad \overline{u} \leq \overline{c} \\ &\kappa_{\min}^R \geq C_4, \quad \kappa_{\min} \geq C_5, \quad |\beta|/\alpha \leq C_6, \quad \eta_0 = 0, \quad \overline{u} > \overline{c}, \end{aligned} \tag{25}$$

where $\kappa_{\min}^R$, $\kappa_{\min}$ *are defined by* (14). *The constants* C_1, C_2, C_3, C_4, C_5, C_6 *are positive and independent of* S. *The functions* $f_{1,3}$ *and* f_2 *are given by* (20) *and* (21).

The proofs of Lemma 2.2 and Lemma 2.3 are given in the Appendix.

The eigenvectors for distinct roots are

$$\psi_1 = r(\kappa_1) = \begin{pmatrix} r_1(\kappa_1) \\ 1 \\ 0 \\ r_4(\kappa_1) \end{pmatrix}, \quad \psi_2 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\psi_3 = r(\kappa_3) = \begin{pmatrix} r_1(\kappa_3) \\ 1 \\ 0 \\ r_4(\kappa_3) \end{pmatrix}, \tag{26}$$

where

$$r_1(\kappa) = \frac{-\overline{\rho}\kappa}{\overline{u}\kappa + S}, \quad r_4(\kappa) = \frac{-(\gamma - 1)M_\infty^2\overline{c}^2\kappa}{\overline{u}\kappa + S - \overline{\varphi}\kappa^2/\overline{\rho}}. \tag{27}$$

A suitable form of the solution close to a possible double root $\kappa_1 = \kappa_3 = \kappa \neq \kappa_2$ is

$$\begin{aligned} \frac{\partial^r \hat{w}_s}{\partial x^r} = &\left(\sigma_1 - \sigma_3\left(\frac{\kappa_3}{\kappa_3 - \kappa_1}\right)\right) r(\kappa_1) \exp\left(\frac{\kappa_1 x}{\varepsilon}\right) \\ &+ \sigma_3\left(\frac{\kappa_3}{\kappa_3 - \kappa_1}\right) r(\kappa_3) \exp\left(\frac{\kappa_3 x}{\varepsilon}\right) + \sigma_2 e_3 \exp\left(\frac{\kappa_2 x}{\varepsilon}\right), \end{aligned} \tag{28}$$

where the subscript s indicates a single root solution.

The form of the solution for a double root is obtained in the following way. The ansatz $\psi = (\psi_1 + x\psi_3)\exp(\kappa x/\varepsilon)$ leads to

$$\begin{aligned} &\{\overline{A}\kappa - \overline{C}\kappa^2 + SI\}\psi_3 = 0 \\ &\{\overline{A}\kappa - \overline{C}\kappa^2 + SI\}\psi_1 = (\varepsilon/\kappa)\{\overline{C}\kappa^2 + SI\}\,\psi_3. \end{aligned}$$

The first equation yields $\psi_3 = \sigma_3(\kappa/\varepsilon)r(\kappa)$, where $r(\kappa)$ is defined by (26)–(27). The second equation yields $\psi_1 = \sigma_1 r(\kappa) +$

$\sigma_3\Lambda(\kappa)r(\kappa)$, where

$$\Lambda(\kappa) = \mathrm{diag}(\lambda_1(\kappa), 0, 0, \lambda_4(\kappa)), \quad \lambda_1(\kappa) = \frac{S}{\bar{u}\kappa + S},$$
$$\lambda_4(\kappa) = \frac{S + \bar{\varphi}\kappa^2/\bar{\rho}}{\bar{u}\kappa + S - \bar{\varphi}\kappa^2/\bar{\rho}}. \tag{29}$$

The solution for a double root $\kappa_1 = \kappa_3 = \kappa \neq \kappa_2$ is

$$\frac{\partial^r \hat{w}_d}{\partial s} = \left\{\sigma_1 r(\kappa) + \sigma_3\Lambda(\kappa)r(\kappa) + \sigma_3 r(\kappa)\left(\frac{\kappa x}{\varepsilon}\right)\right\}\exp\left(\frac{\kappa x}{\varepsilon}\right)$$
$$+ \sigma_2 e_3 \exp\left(\frac{\kappa_2 x}{\varepsilon}\right), \tag{30}$$

where the subscript d indicates a double root solution.

Integration of $\partial^r\hat{w}_s/\partial x^r$ and $\partial^r\hat{w}_d/\partial x^r$ from $-\infty$ to x leads to

$$\hat{w}_s = \left(\frac{\varepsilon}{\kappa_1}\right)^r\left\{\sigma_1 - \sigma_3\left(\frac{\kappa_3}{\kappa_3 - \kappa_1}\right)\right\} r(\kappa_1)\exp\left(\frac{\kappa_1 x}{\varepsilon}\right)$$
$$+ \left(\frac{\varepsilon}{\kappa_3}\right)^r\left\{\sigma_3\left(\frac{\kappa_3}{\kappa_3 - \kappa_1}\right)\right\} r(\kappa_3)\exp\left(\frac{\kappa_3 x}{\varepsilon}\right)$$
$$+ \left(\frac{\varepsilon}{\kappa_2}\right)^r \sigma_2 e_3 \exp\left(\frac{\kappa_2 x}{\varepsilon}\right) \tag{31}$$

$$\hat{w}_d = \left(\frac{\varepsilon}{\kappa}\right)^r\left\{\sigma_1 + \sigma_3\Lambda(\kappa) + \sigma_3\left(\left(\frac{\kappa x}{\varepsilon}\right) - r\right)\right\} r(\kappa)\exp\left(\frac{\kappa x}{\varepsilon}\right)$$
$$+ \left(\frac{\varepsilon}{\kappa_2}\right)^r \sigma_2 e_3 \exp\left(\frac{\kappa_2 x}{\varepsilon}\right). \tag{32}$$

The following lemma will be useful.

LEMMA 2.4. *Consider* (11) *with* $\overline{P}_0 = 0$ *and* $\mathrm{Re}(S) > 0$. *Let* $\hat{w}_s$ *and* $\hat{w}_d$ *be the single and double root solutions given by* (31) *and* (32), *respectively. The solutions satisfy*

$$\lim_{\kappa_3 \to \kappa_1} \hat{w}_s = \hat{w}_d. \tag{33}$$

where κ_1 *and* κ_3 *are the two roots of* $f_{1,3}(\kappa) = 0$ *with positive real parts.*

Proof. The introduction of $\kappa_1 = \kappa$ and $\kappa_3 = \kappa + \delta\kappa$ into (31) gives us

$$\hat{w}_s = \left(\frac{\varepsilon}{\kappa}\right)^r \exp\left(\frac{\kappa x}{\varepsilon}\right)\left\{\left(\sigma_1 - \sigma_3\left(\frac{\kappa + \delta\kappa}{\delta\kappa}\right)\right) r(\kappa)\right.$$
$$\left. + \sigma_3\left(\frac{1}{1 + \delta\kappa/\kappa}\right)^r\left(\frac{\kappa + \delta\kappa}{\delta\kappa}\right) r(\kappa + \delta\kappa)\exp\left(\frac{\delta\kappa x}{\varepsilon}\right)\right\}$$
$$+ \left(\frac{\varepsilon}{\kappa_2}\right)^r \sigma_2 e_3 \exp\left(\frac{\kappa_2 x}{\varepsilon}\right). \tag{34}$$

Equation (34) can be linearised for small $\delta\kappa$. By using

$$(1 + \delta\kappa/\kappa)^{-r} = 1 - r\delta\kappa/\kappa + O(|\delta\kappa|^2)$$
$$\exp(\delta\kappa x/\varepsilon) = 1 + (\delta\kappa x/\varepsilon) + O(|\delta\kappa|^2)$$
$$r(\kappa + \delta\kappa) = r(\kappa) + (\delta\kappa/\kappa)\Lambda(\kappa)r(\kappa) + O(|\delta\kappa|^2),$$

we obtain $\hat{w}_s = \hat{w}_d + O(|\delta\kappa|)$. This concludes the proof of Lemma 2.4.

The distinct root solution given by (28) with the eigenvectors defined by (26)–(27) leads to

$$E = E_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_4(\kappa_1) & 0 & (\kappa_3/(\kappa_3 - \kappa_1))(r_4(\kappa_3) - r_4(\kappa_1)) \end{pmatrix} \tag{35}$$

and

$$\mathrm{Det}(E_s) = \frac{\kappa_3}{\kappa_3 - \kappa_1}(r_4(\kappa_3) - r_4(\kappa_1))$$
$$= \frac{-\kappa_3(\gamma - 1)M_\infty^2\bar{c}^2(S + \bar{\varphi}\kappa_1\kappa_3/\bar{\rho})}{(\bar{u}\kappa_1 + S - \bar{\varphi}\kappa_1^2/\bar{\rho})(\bar{u}\kappa_3 + S - \bar{\varphi}\kappa_3^2/\bar{\rho}}. \tag{36}$$

The double root solution given by (30) leads to

$$E = E_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_4(\kappa) & 0 & r_4(\kappa)\lambda_4(\kappa) \end{pmatrix} \tag{37}$$

and the corresponding determinant becomes

$$\mathrm{Det}(E_d) = r_4(\kappa)\lambda_4(\kappa) = \frac{-\kappa(\gamma - 1)M_\infty^2\bar{c}^2(S + \bar{\varphi}\kappa^2/\bar{\rho}}{(\bar{u}\kappa + S - \bar{\varphi}\kappa^2/\bar{\rho})^2}. \tag{38}$$

Note that the determinant in the single root case goes smoothly over to the determinant in the double root case as $\kappa_3 \to \kappa_1 = \kappa$, i.e.,

$$\mathrm{Det}(E_s) = \mathrm{Det}(E_d) + O(|\kappa_3 - \kappa_1|),$$
$$\lim_{\kappa_3 \to \kappa_1} \mathrm{Det}(E_s) = \mathrm{Det}(E_d). \tag{39}$$

If the system (13) is singular i.e., $\mathrm{Det}(E) = 0$, then (11) cannot be solved for that particular S.

LEMMA 2.5. *Consider* (11) *with* $\overline{P}_0 = 0$ *and* $\mathrm{Re}(S) = \eta \geq 0$. *The matrix* E *in* (13) *satisfies*

$$|\mathrm{Det}|(E)| \geq C_0, \tag{40}$$

where C_0 *is a positive constant independent of* η *and* ε.

The proof of Lemma 2.5 is given in the Appendix. We need an estimate of $\hat{w}$ in terms of $\hat{g}$.

LEMMA 2.6. *Consider* (11) *with* $\overline{P}_0 = 0$ *and* $\mathrm{Re}(S) > 0$. *The solution* $\hat{w}$ *satisfies*

$$|\hat{w}|^2 \leq \text{const} \left(\frac{\varepsilon}{\kappa_{\min}}\right)^{2r} \exp\left(\frac{\kappa_{\min}^R x}{\varepsilon}\right) |\hat{g}|^2, \tag{41}$$

where C_1 *is a positive constant independent of* S, ε, *and* $\kappa_{\min}$, $\kappa_{\min}^R$ *are defined by* (14).

Proof. Let us first consider the case with distinct roots $\kappa_1 \neq \kappa_3$. A direct calculation using the matrix $E = E_s$ from (35) leads to

$$\sigma_1 = \hat{g}_1, \quad \sigma_2 = \hat{g}_2, \quad \sigma_3 = \hat{f}_3(\kappa_1, \kappa_3) = \frac{\hat{g}_3 - r_4(\kappa_1)\hat{g}_1}{\mathrm{Det}(E_s)}. \tag{42}$$

By using (31) and (42) we get

$$\hat{w} = H_1 \left(\frac{\varepsilon}{\kappa_1}\right)^r \exp\left(\frac{\varepsilon}{\kappa_1}\right) + H_2 \left(\frac{\varepsilon}{\kappa_2}\right)^r \exp\left(\frac{\varepsilon}{\kappa_2}\right) + H_3 \left(\frac{\varepsilon}{\kappa_3}\right)^r \exp\left(\frac{\varepsilon}{\kappa_3}\right), \tag{43}$$

where

$$\begin{aligned} H_1 &= G_1 r(\kappa_1), & G_1 &= +\frac{r_4(\kappa_3)}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_1 - \frac{1}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_3 \\ H_2 &= G_2 e_3, & G_2 &= \hat{g}_2 \\ H_3 &= G_3 r(\kappa_3), & G_3 &= -\frac{r_4(\kappa_1)}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_1 + \frac{1}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_3. \end{aligned} \tag{44}$$

The boundedness of the components $r_1(\kappa)$ and $r_4(\kappa)$ for $|S_l| < \infty$ leads to

$$|H_1| \leq \text{const } |\hat{g}|, \quad |H_2| \leq \text{const } |\hat{g}|, \quad |H_3| \leq \text{const } |\hat{g}|. \tag{45}$$

Let us next investigate the case when $|S| \to \infty$. The roots for large S are

$$\begin{aligned} \kappa_1 &= \sqrt{\overline{\rho} S/\overline{\theta}} + \overline{\rho}\overline{u}/2\overline{\theta} + O(|S|^{-1/2}) \\ \kappa_2 &= \sqrt{\overline{\rho} S/\overline{\mu}} + \overline{\rho}\overline{u}/2\overline{\mu} + O(|S|^{-1/2}) \\ \kappa_3 &= \sqrt{\overline{\rho} S/\overline{\varphi}} + \overline{\rho}\overline{u}/2\overline{\varphi} + O(|S|^{-1/2}) \end{aligned}$$

and we obtain

$$\begin{aligned} r_1(\kappa_1) &= O(|S|^{-1/2}), & r_4(\kappa_1) &= O(|S|^{-1/2}) \\ r_1(\kappa_3) &= O(|S|^{-1/2}), & r_4(\kappa_3) &= O(|S|^{+1/2}). \end{aligned} \tag{46}$$

The component $r_4(\kappa_3)$ is unbounded as $|S| \to \infty$. By inserting the estimates (46) into (44) we obtain

$$\lim_{|S|\to\infty} H_1 = \lim_{|S|\to\infty} \begin{pmatrix} \frac{r_1(\kappa_1) r_4(\kappa_3)}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_1 - \frac{r_1(\kappa_1)}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_3 \\ \frac{r_4(\kappa_3)}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_1 - \frac{1}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_3 \\ 0 \\ \frac{r_4(\kappa_1) r_4(\kappa_3)}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_1 - \frac{r_4(\kappa_1)}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{g}_1 \\ 0 \\ 0 \end{pmatrix},$$

$$\lim_{|S|\to\infty} H_3 = \lim_{|S|\to\infty} \begin{pmatrix} -\frac{r_1(\kappa_3) r_4(\kappa_1)}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_1 + \frac{r_1(\kappa_3)}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_3 \\ -\frac{r_4(\kappa_1)}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_1 + \frac{1}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_3 \\ 0 \\ -\frac{r_4(\kappa_1) r_4(\kappa_3)}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_1 + \frac{r_4(\kappa_3)}{r_4(\kappa_3) - r_4(\kappa_1)} \hat{g}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hat{g}_3 \end{pmatrix},$$

and hence the estimate (45) holds for all S with $\mathrm{Re}(S) > 0$ in the case with distinct roots. Equations (43) and (45) lead to (41).

Next, consider the case with a possible double root $\kappa_1 = \kappa_3 = \alpha + i\beta$. A direct calculation using the matrix $E = E_d$ from (37) leads to

$$\sigma_1 = \hat{g}_1, \quad \sigma_2 = \hat{g}_2, \quad \sigma_3 = \hat{f}_3(\kappa, \kappa) = \frac{\hat{g}_3 - r_4(\kappa)\hat{g}_1}{\mathrm{Det}(E_d)}. \tag{47}$$

Note that σ_3 in (42) goes smoothly over to σ_3 given by (47) as $\kappa_3 \to \kappa_1$. The boundedness of the components $r_1(\kappa)$ and $r_4(\kappa)$ in this part of the complex S-plane leads to the boundedness of $\hat{f}_3$. Equations (32) and (47) lead to

$$\hat{w}_d = \left(\frac{\varepsilon}{\kappa}\right)^r \left\{F_1 + F_3\left(\left(\frac{\kappa x}{\varepsilon}\right) - r\right)\right\} r(\kappa) \exp\left(\frac{\kappa x}{\varepsilon}\right) + \left(\frac{\varepsilon}{\kappa_2}\right)^r \{F_2\} e_3 \exp\left(\frac{\kappa_2 x}{\varepsilon}\right),$$

where

$$\begin{aligned} F_1 &= \hat{g}_1 + \hat{f}_3 \Lambda(\kappa), & |F_1| &\leq \text{const } |\hat{g}| \\ F_2 &= \hat{g}_2, & |F_2| &\leq \text{const } |\hat{g}| \\ F_3 &= \hat{f}_3, & |F_3| &\leq \text{const} |\hat{g}|. \end{aligned}$$

The magnitude of $\hat{w}_d$ satisfies

$$|\hat{w}_d|^2 \leq \text{const}\left\{\left|\frac{\varepsilon}{\kappa}\right|^{2r}\{|F_1|^2 + |F_3|^2(F(x)+r)\}\exp\left(\frac{\alpha x}{\varepsilon}\right) + \left|\frac{\varepsilon}{\kappa_2}\right|^{2r}|F_2|^2\exp\left(\frac{\mathrm{Re}(\kappa_2)x}{\varepsilon}\right)\right\},$$

where $F(x) = |\kappa x/\varepsilon|^2 \exp(\alpha x/\varepsilon)$. The solution $\hat{w}_d$ satisfies the estimate (41) if $F(x)$ is bounded. The maximum of $F(x)$ is attained at $x = -2\varepsilon/\alpha$ and $F(-2(\varepsilon/\alpha)) = 4\exp(-2)\{1 + (\beta/\alpha)^2\}$. Lemma 2.2 states that the quotient β/α is bounded for κ and S within the bounds given in (23).

The estimate (41) is valid both for distinct roots and in the case of a double root. Lemma 2.6 follows since (see Lemma 2.4), the single root solution goes smoothly over to the double root solution as $\kappa_3 \to \kappa_1$. This concludes the proof of Lemma 2.6.

Proof of Theorem 2.2. Parseval's relation, the estimate (41), and the estimate (25) for subsonic flow imply

$$\int_0^\infty |w(x,t)|^2\exp(-2\tilde{\eta}t)\,dt \leq c_2(1/\tilde{\eta})^{2r}\exp(\alpha_2\tilde{\eta}x) \int_0^\infty |g(t)|_2\exp(-2\tilde{\eta}t)\,dt. \tag{48}$$

For supersonic flow the result is

$$\int_0^\infty |w(x,t)|^2\exp(-2\tilde{\eta}t)\,dt \leq c_1\varepsilon^{2r}\exp\left(\frac{\alpha_1 x}{\varepsilon}\right) \int_0^\infty |g(t)|^2\exp(-2\tilde{\eta}t)\,dt. \tag{49}$$

The value of $g(t)$ for $t > T$ cannot change $w(x,t)$ for $t \leq T$. By letting $g(t) = 0$ for $t > T$ in (48), (49) the estimates (17), (18) follows.

Let $s = (\tilde{\eta} + i\tilde{\xi})$ be of order one; i.e., $S = \varepsilon s = O(\varepsilon)$. In this part of the complex S-plane there are no double roots and the solution $\hat{w}$ is given by (43), where $|H_i| \geq \text{const}\,|\hat{g}|$, $i = 1, 2, 3$. The eigenvalues κ for small S are (see the proof of Lemma 2.1 in the Appendix)

$$\kappa_1 = \underbrace{\frac{S}{|\bar{u}-\bar{c}|}\{1+O(S)\}}_{O(\varepsilon)},\quad \kappa_2 = \underbrace{\kappa_2^{(0)}\{1+O(S)\}}_{O(1)},$$

$$\kappa_3 = \underbrace{\kappa_3^{(0)}\{1+O(S)\}}_{O(1)},$$

and hence

$$|H_i(\varepsilon/\kappa_i)^r\exp(\kappa_i x/\varepsilon)|\,|H_1(\varepsilon/\kappa_1)^r\exp(\kappa_1 x/\varepsilon)|^{-1} = O\left(|\hat{g}|\varepsilon^r\exp\left(\frac{\kappa_i^{(0)}x}{\varepsilon}\right)\right),\quad i = 2, 3. \tag{51}$$

Parseval's relation and the estimates (50), (51) lead to

$$\int_0^\infty |w(x,t)|^2\exp(-2\tilde{\eta}t)\,dt \geq \frac{1}{2\pi}\int_{-\tilde{\xi}_0}^{+\tilde{\xi}_0}|\hat{w}(x,\tilde{\xi})|^2 d\tilde{\xi} \geq \text{const}\exp\left(\frac{2\tilde{\eta}x}{|\bar{u}-\bar{c}|}\right)\left\{\frac{|\bar{u}-\bar{c}|^2}{\tilde{\eta}^2 + \tilde{\xi}_0^2}\right\}^r\int_{-\tilde{\xi}_0}^{+\tilde{\xi}_0}|\hat{g}|^2\,d\tilde{\xi}.$$

In general $\int_{-\tilde{\xi}_0}^{+\tilde{\xi}_0}|\hat{g}|^2 d\tilde{\xi} \neq 0$ and, hence, the estimate (19) follows. This concludes the proof of Theorem 2.2.

2.4. *The Disturbed Problem with Zero Order Terms*

The influence of the zero order operator $\overline{P}_0$ will now be including the analysis. Consider (12); we have $\mathrm{Det}\{SI - \varepsilon(\overline{P}_0 + \overline{P}_1(\kappa/\varepsilon))\} = f(\kappa)$, where

$$f(\kappa) = f_0(\kappa) + \varepsilon^{2(1-q)}f_1(\kappa),\quad f_1(\kappa) = b_{40}\kappa^4 + b_{30}\kappa^3 + (b_{20} + b_{21}S)\kappa^2 + (b_{11}S)\kappa + b_{02}S^2. \tag{52}$$

The function $f_0(\kappa)$ is given by (22) and

$$\begin{aligned}
b_{40} &= +(\bar{c}^2/\gamma)\{\bar{\theta}\bar{\varphi}(\tilde{\rho}_y/\bar{\rho})^2/\bar{\rho}_2\}\\
b_{30} &= -(\bar{c}^2/\gamma)\{\bar{u}(\bar{\varphi} + 2\bar{\theta})(\tilde{\rho}_y/\bar{\rho})^2/\bar{\rho} - \bar{\varphi}(\tilde{\rho}_y/\bar{\rho})\tilde{u}_y/\bar{\rho}\}\\
b_{21} &= -(\bar{c}^2/\gamma)\{(\bar{\varphi} + 2\bar{\theta})(\tilde{\rho}_y/\bar{\rho})^2/\bar{\rho}\}\\
b_{20} &= +(\bar{c}^2/\gamma)\{(\gamma-2)\bar{u}\tilde{u}_y(\tilde{\rho}_y/\bar{\rho}) + 2(\bar{u}^2 - \bar{c}^2)(\tilde{\rho}_y/\bar{\rho})^2\}\\
b_{11} &= +(\bar{c}^2/\gamma)\{(\gamma-2)\tilde{u}_y(\tilde{\rho}_y/\bar{\rho}) + 4\bar{u}(\tilde{\rho}_y/\bar{\rho})^2\}\\
b_{02} &= +(\bar{c}^2/\gamma)\{2(\tilde{\rho}_y/\bar{\rho})^2\}.
\end{aligned} \tag{53}$$

The constants a_{ik}, b_{ik} in f_0 and $_1$ are of order one.

Let $\tilde{\kappa}$ be a root with positive real part to $f_0 = 0$ and $|S| \geq \delta_1 > 0$ for all $\varepsilon \to 0$. A Taylor expansion of $f(\kappa) = f_0(\kappa) + \varepsilon^{2(1-q)}f_1(\kappa)$ around $\tilde{\kappa}$ leads to

$$\kappa = \tilde{\kappa} + \delta\kappa\varepsilon^{n(1-q)}, \tag{54}$$

where $\delta\kappa = O(1)$ is independent of ε. Depending on the polynomials f_0 and f_1 we get different values of n:

- $f_0(\tilde{\kappa}) = 0$, $\partial f_0(\tilde{\kappa})/\partial\kappa \neq 0$, and $f_1(\tilde{\kappa}) \neq 0$ lead to $\delta\kappa \neq 0$, $n = 2$.

• $f_0(\bar{\kappa}) = 0$, $\partial f_0(\bar{\kappa})/\partial\kappa \neq 0$, and $f_1(\bar{\kappa}) = 0$ lead to $\delta\kappa = 0$.

• $f_0(\bar{\kappa}) = 0$, $\partial f_0(\bar{\kappa})/\partial\kappa = 0$, and $f_1(\bar{\kappa}) \neq 0$ lead to $\delta\kappa \neq 0$, $n = 1$.

• $f_0(\bar{\kappa}) = 0$, $\partial f_0(\bar{\kappa})/\partial\kappa = 0$, and $f_1(\bar{\kappa}) = 0$, $\partial f_1(\bar{\kappa})/\partial\kappa \neq 0$ lead to $\delta\kappa \neq 0$, $n = 2$.

• $f_0(\bar{\kappa}) = 0$, $\partial f_0(\bar{\kappa})/\partial\kappa = 0$, and $f_1(\bar{\kappa}) = 0$, $\partial f_1(\bar{\kappa})/\partial\kappa \neq 0$ lead to $\delta\kappa = 0$.

The corrections of the undisturbed eigenvalues due to the operator $\overline{P}_0$ are small if $|S|$ is strictly greater than zero. However, for small $|S|$, the operator $\overline{P}_0$ will play a significant role.

LEMMA 2.7. *Let ε be sufficiently small and* $\mathrm{Re}(S) = \eta = \varepsilon\tilde{\eta} > 0$. *Then the eigenvalues κ^+ given by* (12) *satisfy*

$$\min_S \kappa_{\min} = \kappa_1 + O(\varepsilon^\delta), \quad \min_S \kappa^R_{\min} = \alpha_1 + O(\varepsilon^\delta), \quad \bar{u} > \bar{c}, \tag{55}$$

$$\min_S \kappa_{\min} = \kappa_2\varepsilon^{1-q} + O(\varepsilon^{1-q+\delta}), \quad \min_S \kappa^R_{\min} = \alpha_2\tilde{\eta}\varepsilon + \alpha_3\varepsilon^{2(1-q)} + O(\varepsilon^{2(1-q)+\delta}), \quad \bar{u} < \bar{c}. \tag{56}$$

The definitions of $\kappa_{\min}$ and $\kappa^R_{\min}$ *are given by* (14). *The constants* κ_1, κ_2, α_1, σ_2, σ_3, *and δ are positive and independent of* S, ε.

To obtain the estimates (55) and (56) the following assumption was made.

Assumption 2.2.

$$\begin{aligned} b_{20} &= +(\bar{c}^2/\gamma)\{(\gamma-2)\bar{u}\tilde{u}_y(\tilde{\rho}_y/\bar{\rho}) + 2(\bar{u}^2 - \bar{c}^2)(\tilde{\rho}_y/\bar{\rho})^2\} > 0, \quad \bar{u} < \bar{c}, \\ b_{20} &= +(\bar{c}^2/\gamma)\{(\gamma-2)\bar{u}\tilde{u}_y(\tilde{\rho}_y/\bar{\rho}) + 2(\bar{u}^2 - \bar{c}^2)(\tilde{\rho}_y/\bar{\rho})^2\} < 0, \quad \bar{u} > \bar{c}. \end{aligned} \tag{57}$$

Condition (57) was checked by using the numerical solutions that will be presented below. It is valid in most of the boundary layers that were calculated, except at a few grid points. The proof of Lemma 2.7 is given in the Appendix.

The following two lemmas correspond to Lemmas 2.5 and 2.6 for the undisturbed problem.

LEMMA 2.8. *Consider* (11) *with* $\mathrm{Re}(S) > 0$. *The matrix E in* (13) *satisfies*

$$|\mathrm{Det}(E)| \geq C_0, \tag{58}$$

where C_0 is a positive constant independent of S, ε.

LEMMA 2.9. *Consider* (11) *with* $\mathrm{Re}(S) > 0$. *The solution $\hat{w}$ satisfies*

$$|\hat{w}|^2 \leq \mathrm{const}\left(\frac{\varepsilon}{\kappa_{\min}}\right)^{2r} \exp\left(\frac{\kappa^R_{\min}x}{\varepsilon}\right)|\hat{g}|^2, \tag{59}$$

where C_1 is a positive constant independent of S, ε, and $\kappa_{\min}$, $\kappa^R_{\min}$ *are defined by* (14).

The proofs of Lemmas 2.8 and 2.9 are given in the Appendix.

Proof of Theorem 2.1. Parseval's relation, the estimate (59), and the estimate (56) for subsonic flow lead directly to

$$\int_0^\infty |w(x,t)|^2 \exp(-2\tilde{\eta}t)\,dt \leq c_2\varepsilon^{2rq}\exp((\alpha_2\tilde{\eta} + \alpha_3\varepsilon^{1-2q})x) \int_0^\infty |g(t)|^2 \exp(-2\tilde{\eta}t)\,dt. \tag{60}$$

For supersonic flow the result is

$$\int_0^\infty |w(x,t)|^2 \exp(-2\tilde{\eta}t)\,dt \leq c_1\varepsilon^{2r}\exp\left(\frac{\alpha_1 x}{\varepsilon}\right) \int_0^\infty |g(t)|^2 \exp(-2\tilde{\eta}t)\,dt. \tag{61}$$

The value of $g(t)$ for $t > T$ cannot change $w(x, t)$ for $t \leq T$. By letting $g(t) = 0$ for $t > T$ in (60), (61) the estimates (15), (16) follow.

3. NUMERICAL EXPERIMENTS

By making computations using the nonlinear Navier–Stokes equations we can check whether the theoretical conclusions drawn from the simplified problem (9) agrees with the results obtained in practice. To obtain the numerical solutions we used a centered finite-volume discretisation in space and the classical fourth-order Runge–Kutta method in time.

3.1. *One-Dimensional Numerical Experiments*

A one-dimensional nonlinear problem is considered in this section (see Fig. 3). The y-gradients on the basic flow are zero, i.e., $\overline{B} = 0$. The nonlinear problem in this section corresponds to the (undisturbed) problem (9) with $\overline{P}_0 = 0$ in the linear analysis above.

Initially the flow field was set to uniform flow $\overline{\phi}_\infty = (\bar{\rho}_\infty, \bar{u}_\infty, \overline{T}_\infty)^{\mathrm{T}}$. At a subsonic inflow boundary ($x = 0$) we used (see [21]),

$$\bar{u} + 2\bar{c}/(\gamma - 1) = \bar{u}_\infty + 2\bar{c}_\infty/(\gamma - 1),$$
$$\overline{T}\bar{\rho}^{1-\gamma} = \overline{T}_\infty\bar{\rho}_\infty^{1-\gamma}, \quad \bar{\theta}\,\bar{u}_x - 2(\bar{k}/\mathrm{Pr})\bar{c}_x = 0$$

as boundary conditions while $\bar{\rho} = \bar{\rho}_\infty$, $\bar{u} = \bar{u}_\infty$, $\overline{T} = \overline{T}_\infty$ was used in the supersonic case. The boundary conditions at the outflow boundary ($x = 1$) were

$$\bar{u}_x = g_1, \quad \overline{T}_x = 0, \quad g_1 = \sin(4\pi t) \tag{62}$$

both for subsonic and supersonic flow.

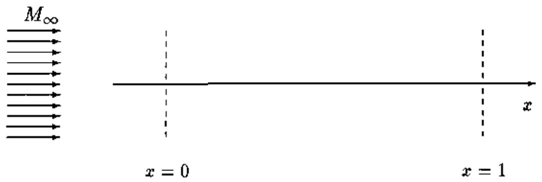

FIG. 3. Geometry definition for the one-dimensional calculations.

The continuous boundary conditions above were implemented using second-order accurate approximations. The density $\overline{\rho}_{N+1}$ at the outflow boundary was obtained using linear extrapolation, i.e., $\overline{\rho}_{N+1} = 2\overline{\rho}_N - \overline{\rho}_{N-1}$ on a uniform mesh.

One calculation using $M_\infty = 0.5$, $\varepsilon = 10^{-3}$ and one calculation using $M_\infty = 1.5$, $\varepsilon = 10^{-3}$ were made. The deviation from uniform flow $|w(x, t)| = |\overline{\phi}(x, t) - \overline{\phi}_\infty|$ and the quotient

$$Q(x, T, \eta) = \frac{\int_0^T |w(x, t)|^2 \exp(-2\eta t)\, dt}{\int_0^T |g(t)|^2 \exp(-2\eta t)\, dt}$$

have been investigated. Theorem 2.2 indicates that the subsonic problem might be ill-posed for derivative boundary conditions in the sense that perturbations of order one in the outflow boundary data might propagate into the computational domain and lead to changes in the solution of order one. This can be realised by choosing η small in (18), (19). The estimate (17) is independent of η and we expect the supersonic problem to well-posed in the sense mentioned above.

The deviation $|w(x, t)|$ (see Fig. 4) due to the oscillating outflow boundary condition spreads into the whole computational domain in the subsonic case and the amplitude is of order one. In the supersonic case the deviation is limited to a very small region close to the outflow boundary and the amplitude is of order ε. The quotient $Q(x, T = 1, \eta)$ (see Fig. 5) is strongly dependent on the parameter η in the subsonic case while the dependency is weak in the supersonic case. Note that the quotient $Q(x, T = 1, \eta = 10)$ is small and has a boundary layer character also in the subsonic case although the computational result is useless. This means that estimates of the form given in Theorem 2.2 and Theorem 2.1 are useful when determining accuracy of the solution only for choices of $\eta = O(1)$.

Finally some remarks about the steady problem are given. It is not possible (see the proof of Theorem 2.2 and [29]) to determine the solution to the steady problem (with $\overline{B} = 0$) uniquely by using derivative boundary conditions at the outflow boundary in the subsonic case. However, by using the time-dependent equations a steady solution can still be computed, but we expect to obtain non-unique solutions at steady state in the subsonic case. The steady problem with derivative boundary

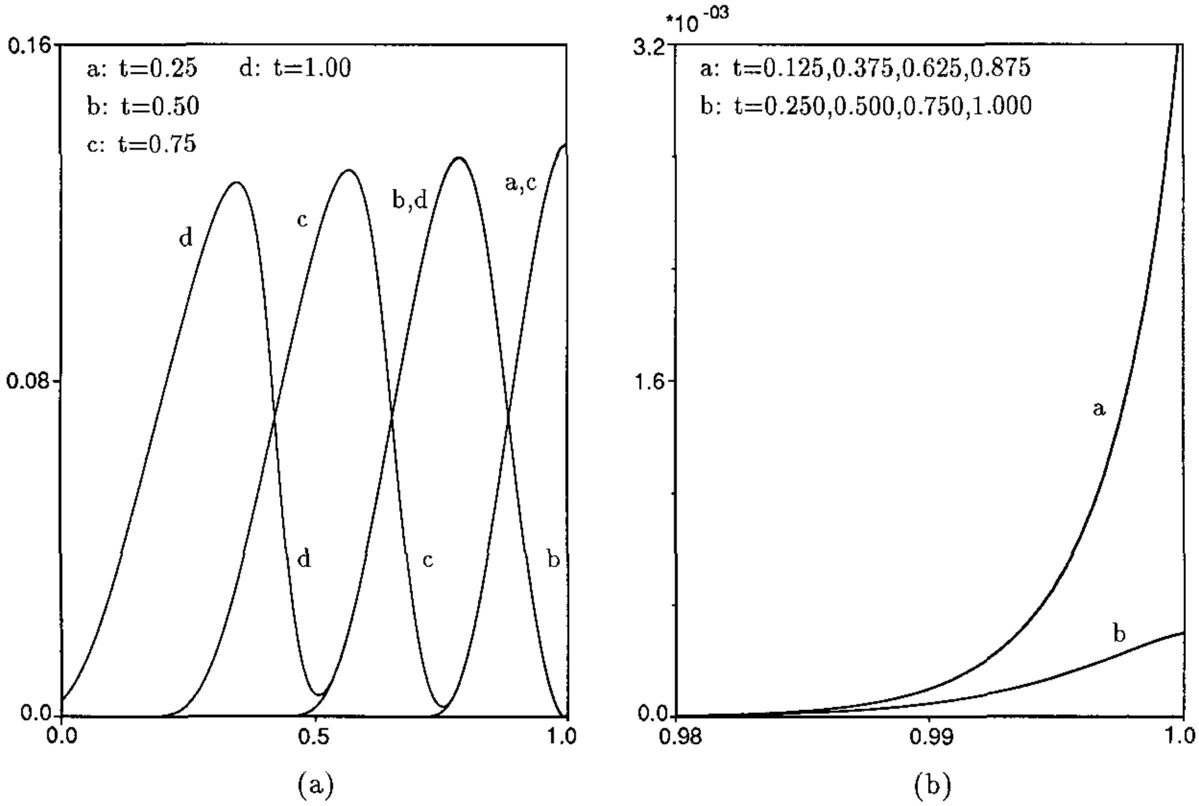

FIG. 4. $|w(x, t)|$: (a) $M_\infty = 0.5$, $\varepsilon = 10^{-3}$; (b) $M_\infty = 1.5$, $\varepsilon = 10^{-3}$.

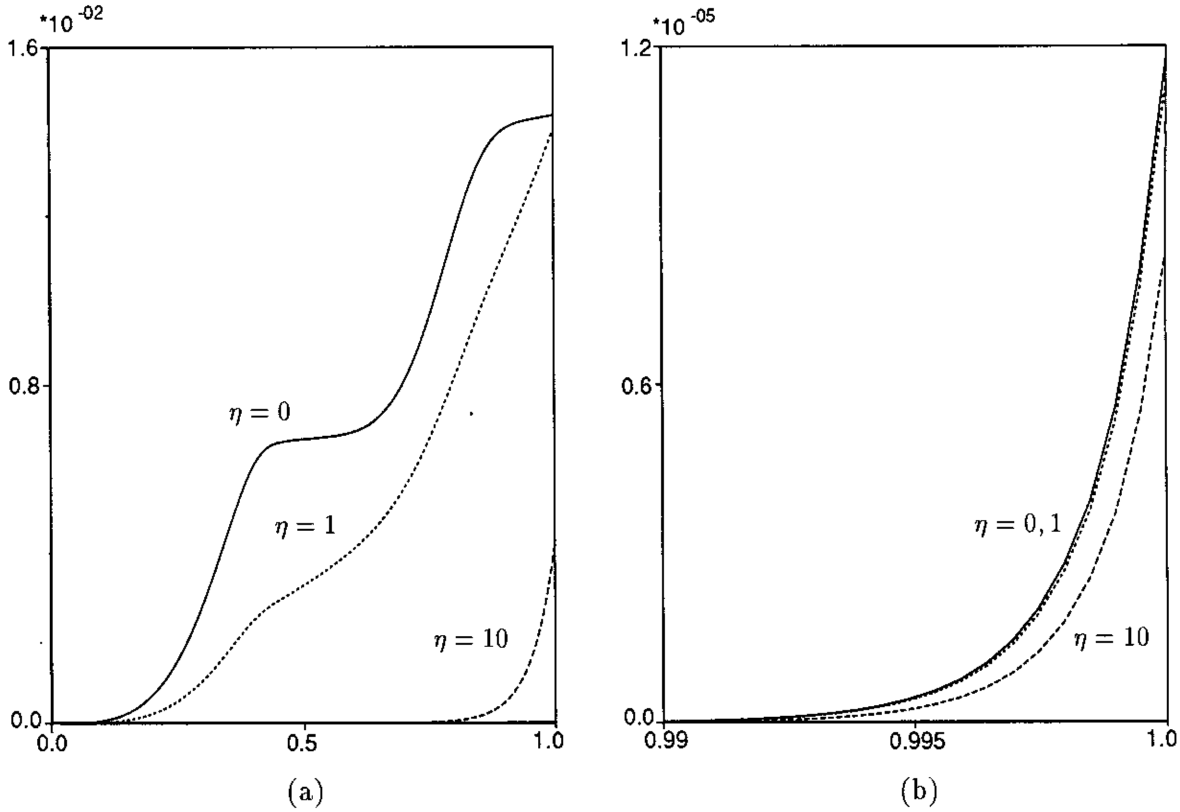

FIG. 5. $Q(x, T = 1, \eta)$: (a) $M_\infty = 0.5$, $\varepsilon = 10^{-3}$; (b) $M_\infty = 1.5$, $\varepsilon = 10^{-3}$.

conditions at the supersonic outflow boundary leads to a unique solution (see the proof of Theorem 2.2 and [29]).

The two cases were integrated up to $t = 1.125$ using the boundary conditions given in (62). At $t = 1.125$ the boundary conditions were changed from $\overline{u}_x = 1$ to $\overline{u}_x = 0$ and the calculations continued. A steady solution was obtained in both cases. The calculation towards steady state is shown in Fig. 6. At steady state we obtained a non-zero deviation $|w(x, \infty)| = |\overline{\phi}(x, \infty) - \overline{\phi}_\infty|$ in the subsonic case while the deviation was zero in the supersonic case. We conclude that the one-dimensional (or undisturbed) time-dependent problem with derivative boundary conditions leads to non-unique steady solutions in the subsonic case and unique steady solutions in the supersonic case.

3.2. *Two-Dimensional Numerical Experiments*

A two-dimensional nonlinear problem, the laminar flow over a flat plate schematically depicted in Fig. 7, with large y-gradients on the basic flow is considered in this section. Large y-gradients on the basic flow implies that $\overline{B} \neq 0$. The nonlinear problem in this section corresponds to the (disturbed) problem (9) with $\overline{P} \neq 0$ in the linear analysis above.

In all the calculations the boundary conditions at the inflow boundary ($x = 0$, $0 \leq y \leq y_m$), the solid wall ($0 \leq x \leq 1$, $y = 0$), and the upper boundary ($0 \leq x \leq 1$, $y = y_m$) were fixed. The boundary conditions used at the solid wall were $\overline{u} = 0$, $\overline{v} = 0$, $\overline{T} = \overline{T}_\infty$. At the subsonic inflow boundary ($x = 0$, $0 \leq y \leq y_i$) we used (see [21]),

$$\overline{u} + 2\overline{c}/(\gamma - 1) = h_1, \quad \overline{T}\overline{\rho}_p^{1-\gamma} = h_2,$$

$$\overline{\theta}\overline{u}_x - 2(\overline{k}/Pr)\overline{c}_x = h_3, \quad \overline{v} = h_4$$

as boundary conditions while at the supersonic inflow boundary ($x = 0$, $y_i \leq y \leq y_m$)

$$\overline{\rho} = r_1, \quad \overline{u} = r_2, \quad \overline{v} = r_3, \quad \overline{T} = r_4$$

were imposed. The functions $h_i(y)$ and $r_i(y)$, $i = 1, 2, 3, 4$, were obtained by solving the compressible boundary layer equations for the flow over a flat plate. At the upper boundary located at $y_m \approx 10\delta$, where δ is the boundary layer thickness, we have accurate data and therefore $\overline{u}_y = 0$, $\overline{v}_y = 0$, $\overline{T}_y = 0$ were used as boundary conditions.

The boundary conditions at the outflow boundary ($x = 1$, $0 \leq y \leq y_m$) were

$$\frac{\partial^r \overline{u}}{\partial x^r} = g_1, \quad \frac{\partial^r \overline{v}}{\partial x^r} = 0, \quad \frac{\partial^r \overline{T}}{\partial x^r} = 0, \quad g_1 = \sin(4\pi t) \qquad (63)$$

both for subsonic and supersonic outflow.

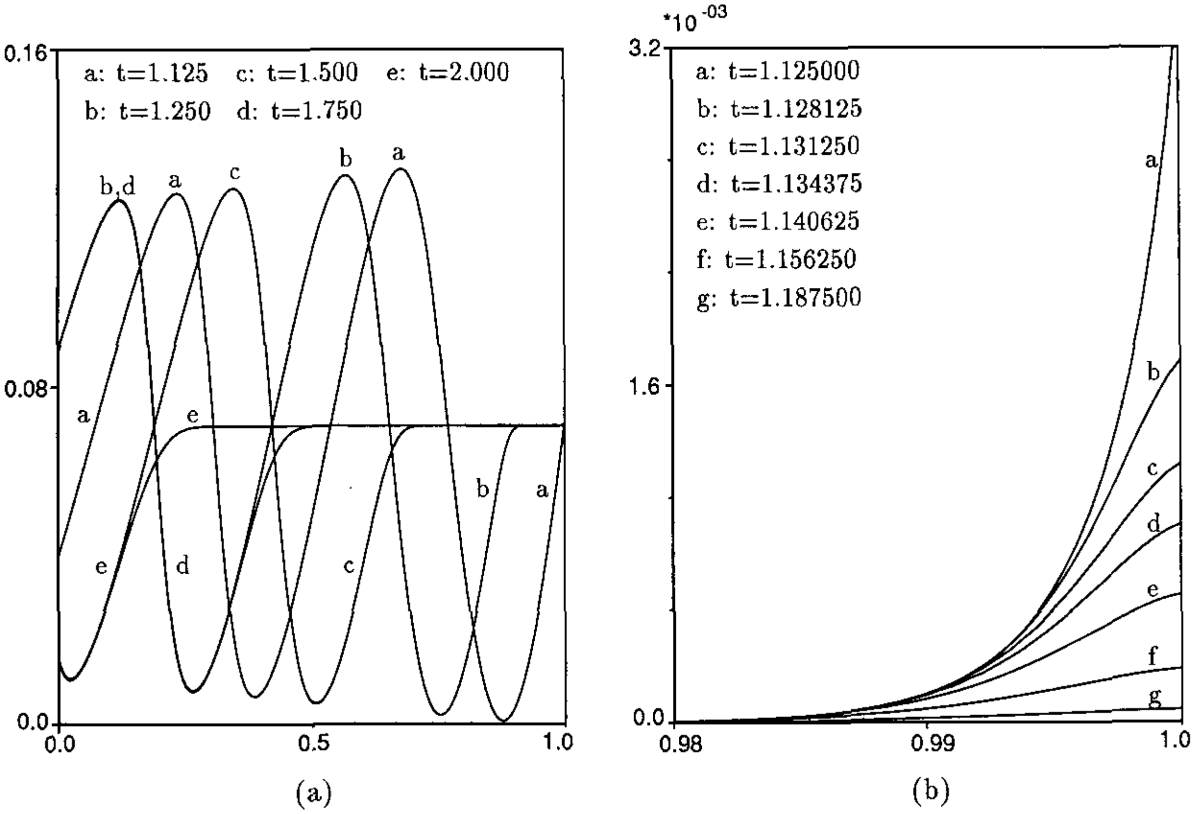

FIG. 6. $|w(x, t)|$ close to steady state: (a) $M_\infty = 0.5$, $\varepsilon = 10^{-3}$; (b) $M_\infty = 1.5$, $\varepsilon = 10^{-3}$.

The continuous boundary conditions above were implemented using second-order accurate approximations. Let the index i, j correspond to the x, y directions. At the upper boundary, the density $\overline{\rho}_{i,M+1}$ was obtained using zero order extrapolation, i.e., $\overline{\rho}_{i,M+1} = \overline{\rho}_{i,M}$. The density $\overline{\rho}_{N+1,j}$ at the outflow boundary was determined using linear extrapolation, i.e., $\overline{\rho}_{N+1,j} = 2\overline{\rho}_{N,j} - \overline{\rho}_{N-1,j}$.

The computations were made at $M_\infty = 2$, $\varepsilon = 10^{-4}$, using first ($r = 1$) and second ($r = 2$) order derivative boundary conditions at the outflow boundary. The time-dependency of the flow was modeled in the following way. First a steady solution $\overline{\phi}_S(x, y)$ using $g_1 = 0$ was computed. Second, the steady solution was used as the initial solution and advanced in time using $g_1 = \sin(4\pi t)$, the time-dependent solution is indicated by $\overline{\phi}_T(x, y, t)$. Finally the deviation $|w(x, t)| = |\overline{\phi}_T(x, \tilde{y}, t) - \overline{\phi}_S(x, \tilde{y})|$ between the solution at a given time and the initial (or steady) solution was studied. The deviation in all figures are evaluated at $y = \tilde{y}$, where $M \approx 0.55$; M is the local Mach number.

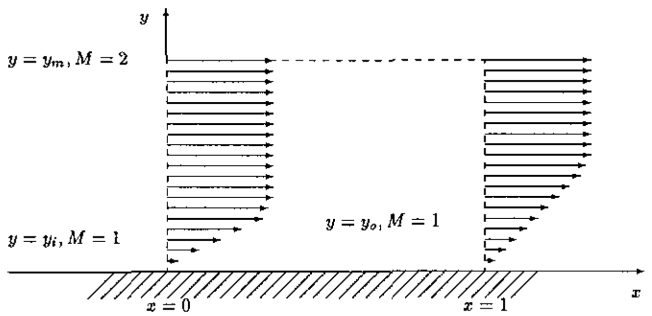

FIG. 7. Geometry definition for the two-dimensional calculations.

Figures 8 and 9 show the deviation $|w(x, t)|$ and the quotient $Q(x, T = 1, \eta)$ for the two cases. The theoretical estimates of the quotient $Q(x = 0, T = 1, \eta)$ (assuming that $q = \frac{1}{2}$) given in Theorem 2.1 and the practical result obtained in the numerical calculations are shown below:

$$Q(x = 0, T = 1, \eta)_{\text{theory}} \propto 10^{-8}, \quad r = 2,$$
$$Q(x = 0, T = 1, \eta)_{\text{calcul}} \propto 10^{-7}, \quad r = 2,$$
$$Q(x = 0, T = 1, \eta)_{\text{theory}} \propto 10^{-4}, \quad r = 1,$$
$$Q(x = 0, T = 1, \eta)_{\text{calcul}} \propto 10^{-3}, \quad r = 1.$$

The theoretical and practical results agree quite well. Furthermore, the exponential decay of the quotient away from the outflow boundary is clearly seen. Note that the result in the

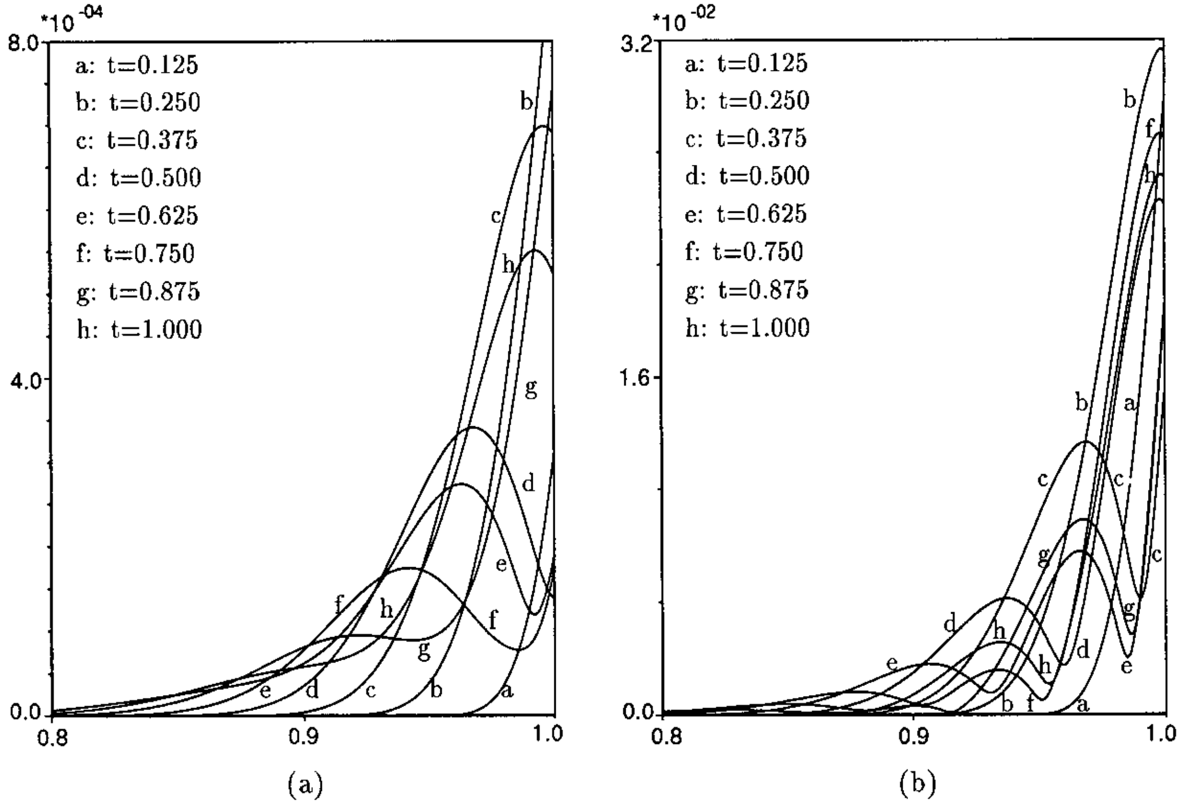

FIG. 8. $|w(x, t)|$: (a) $r = 2$, $\varepsilon = 10^{-4}$; (b) $r = 1$, $\varepsilon = 10^{-4}$.

two-dimensional calculations evaluated at $M \approx 0.55$ has similarities with the one-dimensional supersonic ($M_\infty = 1.5$) calculation; see Figs. 4 and 5.

Finally, just as in the one-dimensional case, we will investigate the question of uniqueness at steady state. It was shown in [29] (see also the proof of Theorem 2.1), that the steady problem (with $\overline{B} \neq 0$) is uniquely determined by using derivative boundary conditions also in the case of subsonic flow. Following the procedure in the one-dimensional problem the two cases were integrated up to $t = 1.125$ using the boundary conditions given in (63). At $t = 1.125$ the boundary conditions were changed from $u_x = 1$ to $u_x = 0$ and the calculations continued. A steady solution was obtained in both cases. In Fig. 10 the calculation towards steady state is shown. At steady state $|w(x, \infty)| = |\overline{\phi}_T(x, \overline{y}, \infty) - \overline{\phi}_S(x, \overline{y})| = 0$ was obtained in both cases. We conclude that the two-dimensional (or disturbed) time-dependent problem with derivative boundary conditions leads to unique steady solutions in both the subsonic and supersonic cases.

4. SUMMARY AND CONCLUSIONS

The use of derivative boundary conditions at artificial outflow boundaries with errors in the boundary data of order one has been investigated. Both the problem when the artificial outflow boundary is located in essentially uniform flow and the situation when the artificial outflow boundary is located in a flow field with large gradients has been investigated.

A linear model problem, where the effect of large gradients in the basic flow field was included, has been derived. These gradients lead to the presence of large zero order terms in the linearised model equations. Appropriate bounds of the solution to the model problem are derived. The bounds become sharper as the order of the derivative boundary conditions increases in the supersonic cases and in the subsonic case with large zero order terms. In the subsonic case without large zero order terms it is shown that a solution of order one is obtained.

In the subsonic outflow case one obtains accurate solutions of the nonlinear problem by using derivative boundary conditions if and only if large transversal gradients are present in the basic flow field. In the supersonic outflow case one always obtains accurate solutions of the nonlinear problem by using derivative boundary conditions. The accuracy does not depend on the presence of large transversal gradients in the basic flow field.

The result of the numerical experiments, where the nonlinear Navier–Stokes equations were used, support the theoretical conclusions drawn from the model problem.

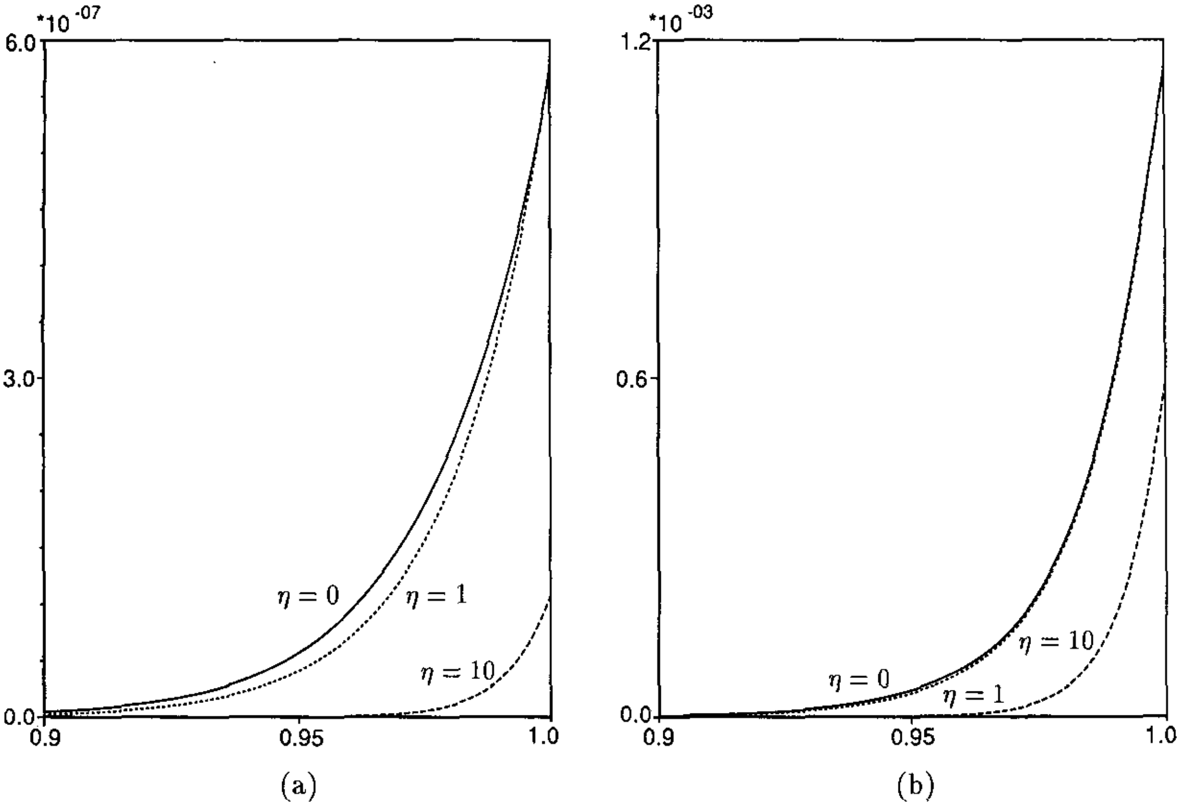

FIG. 9. $Q(x, T = 1, \eta)$: (a) $r = 2$, $\varepsilon = 10^{-4}$; (b) $r = 1$, $\varepsilon = 10^{-4}$.

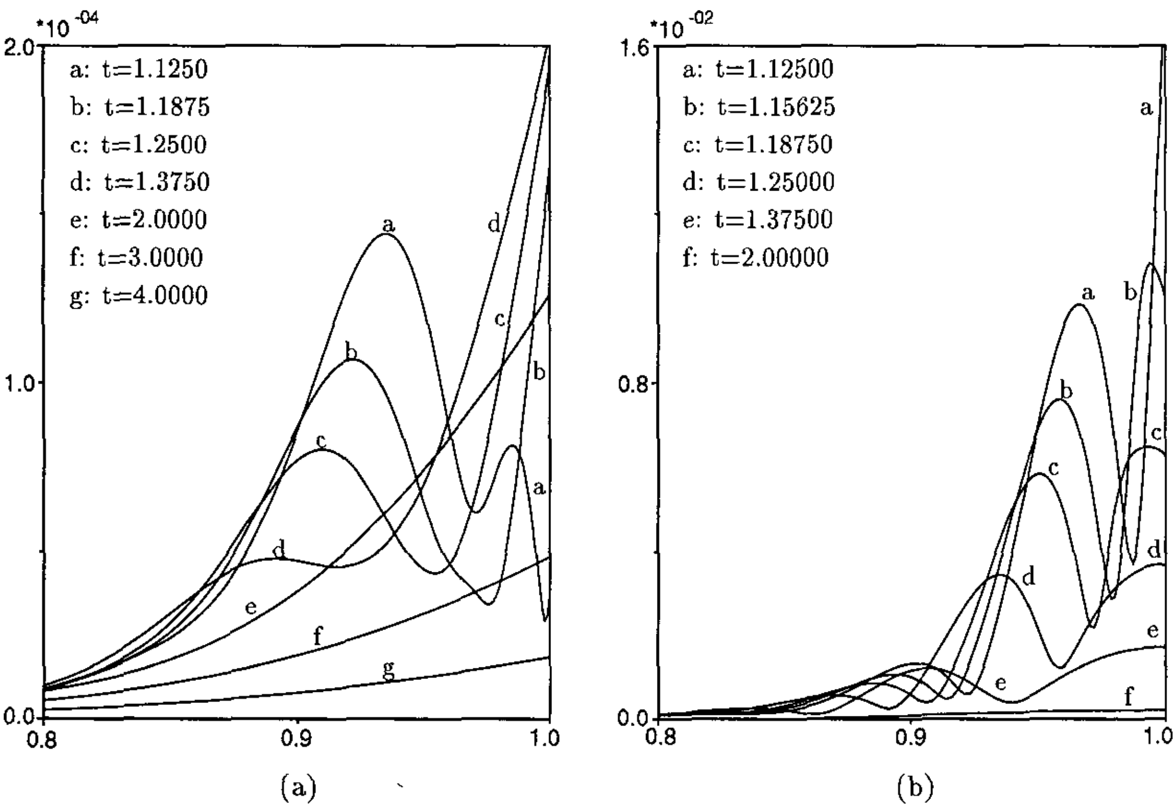

FIG. 10. $|w(x, t)|$ close to steady state: (a) $r = 2$, $\varepsilon = 10^{-4}$; (b) $r = 1$, $\varepsilon = 10^{-4}$.

APPENDIX A

Proof of Lemma 2.1. For $|S| \ll 1$, κ can be expanded in terms of S,

$$\kappa_j = \kappa_j^{(0)} S^{\alpha_j^{(0)}} + \kappa_j^{(1)} S^{\alpha_j^{(1)}} + \ldots, \tag{64}$$

where $\alpha_j^{(i+1)} > \alpha_j^{(i)}$, $\alpha_j^{(0)} \geq 0$. The coefficients $\kappa_j^{(i)}$ and $\alpha_j^{(i)}$ are constants independent of S. The result is

$$\alpha_{1,2,3,4}^{(0)} = 1, \quad \alpha_{1,2,3,4}^{(1)} = 2, \quad \alpha_{5,6,7}^{(0)} = 0, \quad \alpha_{5,6,7}^{(1)} = 1, \tag{65}$$

and

$$\begin{aligned}
\kappa_1^{(0)} &= -1/(\bar{u} + \bar{c}),\\
\kappa_1^{(1)} &= +(1/(\bar{u} + \bar{c}))^3(\bar{\theta} + (\gamma - 1)\bar{\varphi}/\gamma)/2\bar{\rho}\\
\kappa_2^{(0)} &= -1/(\bar{u} - \bar{c}),\\
\kappa_2^{(1)} &= +(1/(\bar{u} - \bar{c}))^3(\bar{\theta} + (\gamma - 1)\bar{\varphi}/\gamma)/2\bar{\rho}\\
\kappa_3^{(0)} &= -1/\bar{u}, \quad \kappa_3^{(1)} = +(1/\bar{u})^3(\bar{\varphi}/\gamma\bar{\rho})\\
\kappa_4^{(0)} &= -1/\bar{u}, \quad \kappa_4^{(1)} = +(1/\bar{u})^3(\bar{\mu}/\bar{\rho})\\
\kappa_5^{(0)} &= \bar{\rho}\bar{u}/\bar{\mu}, \quad \kappa_5^{(1)} = +1/\bar{u},
\end{aligned} \tag{66}$$

$$\kappa_{6,7}^{(0)} = \frac{\bar{\rho}(\bar{u}^2(\bar{\theta} + \bar{\varphi}) - \bar{c}^2\bar{\varphi}/\gamma)}{2\bar{u}\bar{\theta}\bar{\varphi}} \pm \sqrt{(\bar{\rho}(\bar{u}^2(\bar{\theta} + \bar{\varphi}) - \bar{c}^2\bar{\varphi}/\gamma)/2\bar{u}\bar{\theta}\bar{\varphi})^2 - \bar{\rho}^2(\bar{u}^2 - \bar{c}^2)/\bar{\theta}\bar{\varphi}} \tag{67}$$

$$\kappa_{6,7}^{(1)} = \frac{\bar{\rho}(\bar{u}^2(\bar{\theta} + \bar{\varphi}) + \bar{c}^2\bar{\varphi}/\gamma)\kappa_{6,7}^{(0)} - 2\bar{\rho}^2\bar{u}^3}{\bar{u}^2(\bar{\theta}\bar{\varphi}\kappa_{6,7}^{(0)2} - (\bar{u}^2 - \bar{c}^2))}. \tag{68}$$

Subsonic flow leads to $\mathrm{Re}(\kappa_7^{(0)}) < 0$ while $\mathrm{Re}(\kappa_7^{(0)}) > 0$ in supersonic flow. The three eigenvalues with positive real part in the subsonic case are $\kappa_1^+ = \kappa_2$, $\kappa_2^+ = \kappa_5$, $\kappa_3^+ = \kappa_6$. In the supersonic case we get $\kappa_1^+ = \kappa_7$, $\kappa_2^+ = \kappa_5$, $\kappa_3^+ = \kappa_6$.

For $|S| \gg 1$ we make the ansatz

$$\kappa_j = \kappa_j^{(0)} S^{\alpha_j^{(0)}} + \kappa_j^{(1)} S^{\alpha_j^{(1)}} + \ldots, \tag{69}$$

where $\alpha_j^{(i+1)} < \alpha_j^{(i)}$. The coefficients $\kappa_j^{(i)}$ and $\alpha_j^{(i)}$ are constants independent of S. The result is

$$\alpha_1^{(0)} = 1, \quad \alpha_1^{(1)} = 0, \quad \alpha_{2,3,4,5,6,7}^{(0)} = \tfrac{1}{2}, \quad \alpha_{2,3,4,5,6,7}^{(1)} = 0, \tag{70}$$

$$\begin{aligned}
\kappa_1^{(0)} &= -1/\bar{u}, & \kappa_1^{(1)} &= -\bar{c}^2/(\gamma\bar{u}\bar{\theta})\\
\kappa_2^{(0)} &= +\sqrt{\bar{\rho}/\bar{\varphi}}, & \kappa_2^{(1)} &= +\bar{\rho}\bar{u}/2\bar{\varphi}\\
\kappa_3^{(0)} &= -\sqrt{\bar{\rho}/\bar{\varphi}}, & \kappa_3^{(1)} &= +\bar{\rho}\bar{u}/2\bar{\varphi}\\
\kappa_4^{(0)} &= +\sqrt{\bar{\rho}/\bar{\theta}}, & \kappa_4^{(1)} &= +\bar{\rho}\bar{u}/2\bar{\theta}\\
\kappa_5^{(0)} &= -\sqrt{\bar{\rho}/\bar{\theta}}, & \kappa_5^{(1)} &= +\bar{\rho}\bar{u}/2\bar{\theta}\\
\kappa_6^{(0)} &= +\sqrt{\bar{\rho}/\bar{\mu}}, & \kappa_6^{(1)} &= +\bar{\rho}\bar{u}/2\bar{\mu}\\
\kappa_7^{(0)} &= -\sqrt{\bar{\rho}/\bar{\mu}}, & \kappa_7^{(1)} &= +\bar{\rho}\bar{u}/2\bar{\mu}.
\end{aligned} \tag{71}$$

Independent of supersonic or subsonic flow we obtain $\kappa_1^+ = \kappa_2$, $\kappa_2^+ = \kappa_6$, $\kappa_3^+ = \kappa_4$. It is easily verified that the leading terms of κ^+ do not coincide for small or large S and, hence, if the double root exists, then $\delta_0 < |S^*| < \delta_1$ which leads to $\delta_2 < |\kappa^+| < \delta_3$. Only the leading terms of κ^+ were necessary in the proof; the other terms were included for completeness.

Proof of Lemma 2.2. Let $S = R\exp(i\Theta)$, where $R \gg 1$ and $-\pi/2 \leq \Theta \leq +\pi/2$. The approximation of κ^+ given by (69), (70), and (71) leads to

$$\begin{aligned}
&\lim_{R\to\infty} \alpha_j = +\sqrt{\bar{\rho}R/\bar{\chi}_j}\cos(\Theta/2),\\
&\qquad -\pi/2 \leq \Theta \leq +\pi/2, j = 1, 2, 3,\\
&\lim_{R\to\infty} \beta_j = +\sqrt{\bar{\rho}R/\bar{\chi}_j}\sin(\Theta/2),\\
&\qquad -\pi/2 \leq \Theta \leq +\pi/2, j = 1, 2, 3,\\
&\lim_{R\to\infty} \beta_j/\alpha_j = \tan(\Theta/2),\\
&\qquad -\pi/2 \leq \Theta \leq +\pi/2, j = 1, 2, 3,
\end{aligned} \tag{72}$$

where $(\bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3) = (\bar{\varphi}, \bar{\theta}, \bar{\mu})$.

Now the situation on the imaginary axis will be investigated. Let $S = i\xi$ and assume that $\kappa = i\beta$ is purely imaginary; this leads to $f_{1,3}(i\beta) = f_{1,3}^R + if_{1,3}^I$, where

$$\begin{aligned}
f_{1,3}^R &= \beta^2\{(\bar{\theta} + \bar{\varphi})(\bar{u}\beta + \xi)^2 - \bar{c}^2\bar{\varphi}\beta^2/\gamma\}/\bar{\rho}\\
f_{1,3}^I &= (\bar{u}\beta + \xi)\{\bar{\theta}\bar{\varphi}\beta^4/\bar{\rho}^2 + \bar{c}^2\beta^2 - (\bar{u}\beta + \xi)^2\}.
\end{aligned}$$

For $\beta^2 = \gamma(\bar{\theta} + \bar{\varphi})(\bar{u}\beta + \xi)^2/(\bar{\varphi}\bar{c}^2)$ we obtain $f_{1,3}^R = 0$ and $f_{1,3}^I = (\bar{u}\beta + \xi)^3 \times g$, where

$$g = \gamma^2\bar{\theta}\bar{\varphi}(\bar{\theta} + \bar{\varphi})^2(\bar{u}\beta + \xi)^2/(\bar{\varphi}^2\bar{c}^4) + \gamma(\bar{\theta} + (\gamma - 1)\bar{\varphi}/\gamma)/\bar{\varphi}.$$

It is obvious that $g \neq 0$ and, hence, $f_{1,3}(i\beta) \neq 0$. Similarly, one can also show that $f_2(i\beta) \neq 0$ and, consequently, $\alpha \geq \delta_1 > 0$ for $S = i\xi$, $|\xi| > \delta_0$. Let us next investigate the quotient $|\beta|/\alpha$. The eigenvalues κ^+ are functions of the coefficients in $f_{1,3}$, f_2, which in turn are functions of S. Since $\alpha \neq 0$ the only possibility for $|\beta|/\alpha$ to be unbounded is that $|\beta| \to \infty$ faster than α when $|\xi| \to \infty$. By using (72) with $\Theta = \pm\pi/2$ it follows that the condition (24) is valid on the imaginary axis for $|\xi| > \delta_0 > 0$.

Close to $S = 0$ the expansion of κ defined by (64), (65), (66), and (67) is valid. Let $S = \eta + i\xi$, $\eta \geq 0$, $|S| = \delta_0$, where δ_0 is sufficiently small. In the subsonic case we get

$$\begin{aligned}
\kappa_1^+ &= \phantom{\kappa_5^{(0)}} + \kappa_2^{(0)}S + \kappa_2^{(1)}S^2 + O(\delta_0^3)\\
\kappa_2^+ &= \kappa_5^{(0)} + \kappa_5^{(1)}S + O(\delta_0^2)\\
\kappa_3^+ &= \kappa_6^{(0)} + \kappa_6^{(1)}S + O(\delta_0^2)
\end{aligned} \tag{73}$$

while supersonic flow leads to

$$\begin{aligned}\kappa_1^+ &= \kappa_7^{(0)} + \kappa_7^{(1)}S + O(\delta_0^2)\\ \kappa_2^+ &= \kappa_5^{(0)} + \kappa_5^{(1)}S + O(\delta_0^2)\\ \kappa_3^+ &= \kappa_6^{(0)} + \kappa_6^{(1)}S + O(\delta_0^2).\end{aligned} \tag{74}$$

The notations in (73) and (74) are those given in (64), (65), (66), and (67). Condition (24) is trivial in the supersonic case since the leading term of κ^+ is a real constant. Note that $\delta_0 = 0$ can be used; i.e., the inequalities hold for all S with $\mathrm{Re}(S) \geq 0$.

In the subsonic case we get

$$\kappa_1^+(S) = \frac{-S}{(\bar{u} - \bar{c})} + \frac{(\bar{\theta} + (\gamma - 1)\bar{\varphi}/\gamma)}{2\bar{\rho}(\bar{u} - \bar{c})^3} S^2 + O(\delta_0^3)$$

and, consequently,

$$\alpha(\eta, \xi) = \frac{-\eta}{(\bar{u} - \bar{c})} + \frac{(\bar{\theta} + (\gamma - 1)\bar{\varphi}/\gamma)}{2\bar{\rho}(\bar{u} - \bar{c})^3}(\eta^2 - \xi^2) + O(\delta_0^3) \tag{75}$$

$$\beta(\eta, \xi) = \frac{-\xi}{(\bar{u} - \bar{c})} + \frac{(\bar{\theta} + (\gamma - 1)\bar{\varphi}/\gamma)}{2\bar{\rho}(\bar{u} - \bar{c})^3}(2\eta\xi) + O(\delta_0^3). \tag{76}$$

Equations (75)–(76) lead to

$$\begin{aligned}\min \alpha &= \lim_{\eta \to 0} \alpha = -\kappa_2^{(1)}\delta_0^2 + O(\delta_0^3) > 0,\\ \max|\beta| &= \lim_{\eta \to 0} |\beta| = +\kappa_2^{(0)}\delta_0 + O(\delta_0^2) > 0.\end{aligned} \tag{77}$$

Condition (24) for subsonic flow follows from (77) with $\delta_0 > 0$. This concludes the proof of Lemma 2.2.

Proof of Lemma 2.3. Condition (25) in the supersonic case follows directly from Lemma 2.2 and we turn to the subsonic case. Let $S = \eta_0 + i\xi$, $\eta_0 > 0$. Equations (75)–(76) are valid close to $S = 0$ and lead to

$$\alpha(\eta_0, 0) > \kappa_2^{(0)}\eta_0, \quad \beta(\eta_0, 0) = O(\eta_0^3), \quad (|\beta|/\alpha)(\eta_0, 0) = O(\eta_0^2).$$

Equations (75)–(76) also show that α and $|\beta|$ are growing functions of ξ for $|S|$ sufficiently small. Lemma 2.2 rules out the possibility of an imaginary κ for $S = i\xi$, $|\xi| \neq 0$. By choosing η_0 sufficiently small, Lemma 2.2 can be used to exclude the possibility of a decreasing α as $|\xi|$ increases. Finally, (72) can be used once more to show that $|\beta|/\alpha$ is bounded as $|\xi| \to \infty$ and, hence, the estimate (25) is valid also for subsonic flow. This concludes the proof of Lemma 2.3.

Proof of Lemma 2.5. The condition for a singular matrix E in both the single and double root cases becomes (see (36) and (38)),

$$\kappa_1\kappa_3 = -S\bar{\rho}/\bar{\varphi}. \tag{78}$$

The relations between the five roots of $f_{1,3}(\kappa) = 0$ can be written

$$\begin{aligned}&(\kappa_1 + \kappa_3) + (\kappa_2 + \kappa_4 + \kappa_5) = -A_4,\\ &(\kappa_1 + \kappa_3)(\kappa_2 + \kappa_4 + \kappa_5) + (\kappa_1\kappa_3) + (\kappa_2\kappa_4 + \kappa_2\kappa_5 + \kappa_4\kappa_5) = +A_3,\\ &(\kappa_1\kappa_3)(\kappa_2 + \kappa_4 + \kappa_5) + (\kappa_1 + \kappa_3)(\kappa_2\kappa_4 + \kappa_2\kappa_5 + \kappa_4\kappa_5) + (\kappa_2\kappa_4\kappa_5) = -A_2,\\ &(\kappa_1\kappa_3)(\kappa_2\kappa_4 + \kappa_2\kappa_5 + \kappa_4\kappa_5) + (\kappa_1 + \kappa_3)(\kappa_2\kappa_4\kappa_5) = +A_1,\\ &(\kappa_1\kappa_3)(\kappa_2\kappa_4\kappa_5) = -A_0,\end{aligned} \tag{79}$$

where, see (20), the coefficients are

$$\begin{aligned}A_4 &= (S\bar{\theta}\,\bar{\varphi} - \bar{u}^2(\bar{\theta} + \bar{\varphi})/\bar{\rho} + \bar{c}^2\bar{\varphi}/\gamma\bar{\rho})/(\bar{u}\bar{\theta}\,\bar{\varphi}/\bar{\rho}^2)\\ A_3 &= (\bar{u}(\bar{u}^2 - \bar{c}^2) - 2\bar{u}S(\bar{\theta} + \bar{\varphi})/\bar{\rho})/(\bar{u}\bar{\theta}\,\bar{\varphi}/\bar{\rho}^2)\\ A_2 &= ((3\bar{u}^2 - \bar{c}^2)S - S^2(\bar{\theta} + \bar{\varphi})/(\bar{\rho})/(\bar{u}\bar{\theta}\bar{\varphi}/\bar{\rho}^2)\\ A_1 &= (3\bar{u}S^2)/(\bar{u}\bar{\theta}\bar{\varphi}/\bar{\rho}^2)\\ A_0 &= (S^3)/(\bar{u}\bar{\theta}\,\bar{\varphi}/\bar{\rho}^2).\end{aligned}$$

By solving for $(\kappa_2\kappa_4\kappa_5)$, $(\kappa_2 + \kappa_4 + \kappa_5)$, and $(\kappa_2\kappa_4 + \kappa_2\kappa_5 + \kappa_4\kappa_5)$ in (79) the two equations for $x = (\kappa_1 + \kappa_3)$ and $y = (\kappa_1\kappa_3)$,

$$\begin{aligned}x^3 + x^2\{A_4\} + x\{A_3 - 2y\} + \{-A_0/y + A_2 - A_4y\} &= 0\\ + x^2 + x\{A_4 - A_0/y^2\} + \{-A_1/y + A_3 - y\} &= 0,\end{aligned}$$

can be derived. The first equation minus x times the second equation leads to

$$x^2\{A_0/y^2\} + x\{A_1/y - y\} + \{-A_0/y + A_2 - A_4y\} = 0. \tag{80}$$

Condition (78) inserted into (80) leads to

$$x_{1,2} = (\kappa_1 + \kappa_3)_{1,2} = (\bar{\rho}/2\bar{\varphi}^2)\{\bar{u}(3\bar{\varphi} - \bar{\theta}) \pm \sqrt{\bar{u}^2(\varphi - \theta)^2 + 4(\gamma - 1)\bar{c}^2\bar{\varphi}^2/\gamma}\}.$$

This means that $\kappa_1 + \kappa_3$ is independent of S and purely real. The eigenvalues must be of the form $\kappa_1 = \alpha_1 + i\beta$ and $\kappa_3 = \alpha_3 - i\beta$. Now since $\alpha_1 > 0$, $\alpha_3 > 0$ we get $\mathrm{Re}(\kappa_1\kappa_3) = \alpha_1\alpha_3 + \beta^2 > 0$ which contradicts condition (78), since $\mathrm{Re}(S) > 0$.

Proof of Lemma 2.7. We have to investigate the eigenvalues for small S. Let S have the form

$$S = \tilde{S}\varepsilon^{\beta}, \quad \beta > 0, \quad \delta_0 < |\tilde{S}| < \delta_1,$$

where δ_0, δ_1 are constants. For κ we make the ansatz

$$\kappa_j = \kappa_j^{(0)} \varepsilon^{\alpha_j^0} + \kappa_j^{(1)} \varepsilon^{\alpha_j^1} + \ldots,$$
$$\alpha_j^i = \alpha_j^i(\beta), \quad \alpha_j^{(i+1)} > \alpha_j^{(i)}, \quad \delta_2 < |\kappa_j^{(i)}| < \delta_3, \quad \delta > 0,$$

where δ_2, δ_3 are constants. Three different cases with decreasing β will be investigated. The magnitudes of κ in the three different cases are

$$\text{i. } \beta = 1 - q + \delta \Rightarrow \alpha_{1,2} = 1 - q + \delta$$
$$\alpha_{3,4} = 1 - q, \qquad \alpha_{5,6,7} = 0$$
$$\text{ii. } \beta = 1 - q \quad \Rightarrow \quad \alpha_{1,2,3,4} = 1 - q, \qquad \alpha_{5,6,7} = 0$$
$$\text{iii. } \beta = 1 - q - \delta \Rightarrow \quad \alpha_{1,2,3,4} = 1 - q - \delta, \quad \alpha_{5,6,7} = 0$$

where $\delta > 0$.

The supersonic case is analysed first. The leading coefficients $\kappa_{5,6,7}^{(0)}$ in all the three cases, i–iii, are given by

$$a_{70}\kappa^3 + a_{60}\kappa^2 + a_{50}\kappa + a_{40} = 0.$$

We have $\mathrm{Re}(\kappa_5^{(0)}) > 0$, $\mathrm{Re}(\kappa_6^{(0)}) > 0$, and $\mathrm{Re}(\kappa_7^{(0)}) > 0$. The coefficients $\kappa_{5,6,7}^{(0)}$ are explicitly given by (66) and (67). By choosing $\kappa_1^+ = \kappa_7$, $\kappa_2^+ = \kappa_5$, $\kappa_3^+ = \kappa_6$ the following estimate is obtained:

$$\bar{u} > \bar{c}, \quad \text{cases i–iii. } \min_j \mathrm{Re}(\kappa_j^+) \geq C_1, \quad \min_j |\kappa_j^+| \geq C_2. \tag{81}$$

The positive constants C_1 and C_2 are independent of ε.

The case with subsonic flow is more complicated. In all the cases i–iii we have $\mathrm{Re}(\kappa_5^{(0)}) > 0$, $\mathrm{Re}(\kappa_6^{(0)}) > 0$ while $\mathrm{Re}(\kappa_7^{(0)}) < 0$. A third eigenvalue with positive real part is necessary.

Consider case i. The equations for remaining leading coefficients are

$$\begin{aligned} b_{20}\kappa^2 + b_{11}\tilde{S}\kappa + b_{02}\tilde{S}^2 = 0 \\ a_{40}\kappa^2 + b_{20} = 0 \end{aligned} \tag{82}$$

which lead to

$$\begin{aligned} &\kappa_1^{(0)} = -C_1\tilde{S}, \quad \kappa_2^{(0)} = -C_2\tilde{S}, \\ &\kappa_3^{(0)} = +\sqrt{-b_{20}/a_{40}}, \quad \kappa_4^{(0)} = -\sqrt{-b_{20}/a_{40}}, \end{aligned} \tag{83}$$

where

$$C_{1,2} = +(\mathrm{b}_{11}/2b_{20}) \pm \sqrt{(b_{11}/2b_{20})^2 - b_{02}/b_{20}}. \tag{84}$$

Condition (57) in Assumption 2.2 leads to three eigenvalues with positive real part, $\kappa_1^+ = \kappa_3$, $\kappa_2^+ = \kappa_5$, $\kappa_3^+ = \kappa_6$. Furthermore, (53) and Assumption 2.2 lead to

$$b_{02} > 0, \quad b_{11} > 0, \quad b_{20} = \bar{u}b_{11} - (\bar{u}^2 + \bar{c}^2)b_{02} > 0. \tag{85}$$

Equation (85) shows that

$$b_{11}^2 - 4b_{20}b_{02} = (b_{11} - 2\bar{u}b_{02})^2 + 4\bar{c}^2 b_{02} \geq 0$$

and, hence, $C_{1,2}$ in (84) are positive and real constants. From (82), (83), and (57) the estimate

$$\bar{u} \leq \bar{c}, \quad \text{case i. } \min_j \mathrm{Re}(\kappa_j^+) \geq C_1\varepsilon^{1-q}, \quad \min_j |\kappa_j^+| \geq C_2\varepsilon^{1-q} \tag{86}$$

is obtained. The constants C_1 and C_2 are positive and independent of ε.

Consider case ii. The equation for remaining leading coefficients is

$$\begin{aligned} &a_{40}\kappa^4 + (a_{31}\tilde{S})\kappa^3 + (a_{22}\tilde{S}^2 + b_{20})\kappa^2 \\ &\quad + (a_{13}\tilde{S}^3 + b_{11}\tilde{S})\kappa + (a_{04}\tilde{S}^4 + b_{02}\tilde{S}^2) = 0. \end{aligned} \tag{87}$$

An explicit solution for $\kappa^{(0)}$ from (87) cannot be obtained. An expansion of the solution in terms of small values of $\bar{u} - \bar{c}$ leads to

$$\kappa^{(0)} = \frac{-\tilde{S}}{(\bar{u} - \bar{c})}\left\{1 - \frac{b_{20}}{(2\tilde{S}^2\bar{c}^3)}(\bar{u} - \bar{c}) + O((\bar{u} - \bar{c})^2)\right\}. \tag{88}$$

The real part of $\kappa^{(0)}$ is zero if $\mathrm{Re}(S) = 0$. This leads us to include the next term in the expansion of κ. The first terms in the expansion of the third eigenvalue with positive real part are

$$\kappa^+ = \kappa^{(0)}\varepsilon^{1-q} + \kappa^{(1)}\varepsilon^{2(1-q)} + O(\varepsilon^{2(1-q)+\delta}). \tag{89}$$

The approximation (88) of $\kappa^{(0)}$ leads to

$$\kappa^{(1)} = \frac{\tilde{S}^2}{(\bar{u} - \bar{c})^3}\left(\frac{\bar{\theta} + \frac{(\gamma - 1)}{\gamma}\bar{\varphi}}{2\bar{p}}\right)\left\{1 + \frac{b_{20}}{(2\tilde{S}^2\bar{c}^3)}\left(\frac{\bar{u}}{\bar{c}}\right)(\bar{u} - \bar{c}) + O((\bar{u} - \bar{c})^2)\right\}. \tag{90}$$

For small values of $|\bar{u} - \bar{c}|$ an approximation of the third eigenvalue in the right half of the complex plane is obtained. We do not expect the variation of $|\bar{u} - \bar{c}|$ to change the character of this eigenvalue and, hence, the eigenvalues with positive real part in case ii are $\kappa_1^+ = \kappa^+$, $\kappa_2^+ = \kappa_5$, $\kappa_3^+ = \kappa_6$. Equations

(89), (88), and (90) lead to the estimate

$$\bar{u} < \bar{c}, \quad \text{case ii.} \min_j \operatorname{Re}(\kappa_j^+) \geq C_1 \tilde{\eta} \varepsilon + C_2 \varepsilon^{2(1-q)}, \quad \min_j |\kappa_j^+| \geq C_3 \varepsilon^{1-q}. \tag{91}$$

The positive constants C_1, C_2, and C_3 are independent of ε.

Consider case iii. The equation for remaining leading coefficients is

$$a_{40}\kappa^4 + (a_{31}\tilde{S})\kappa^3 + (a_{22}\tilde{S}^2)\kappa^2 + (a_{13}\tilde{S}^3)\kappa + (a_{04}\tilde{S}^4) = 0. \tag{92}$$

The real part of the leading coefficient $\kappa^{(0)} = -\tilde{S}/(\bar{u} - \bar{c})$ is zero if $\operatorname{Re}(S) = 0$; this leads us to include more terms in the expansion of κ. The third eigenvalue with positive real part can be written

$$\kappa^+ = \kappa^{(0)}\varepsilon^{1-q-\delta} + \kappa^{(1)}\varepsilon^{1-q+\delta} + \kappa^{(2)}\varepsilon^{2(1-q-\delta)} + O(\varepsilon^{\gamma_1}), \quad \delta < (1-q)/2, \tag{93}$$

$$\kappa^+ = \kappa^{(0)}\varepsilon^{1-q-\delta} + \kappa^{(2)}\varepsilon^{2(1-q-\delta)} + O(\varepsilon^{\gamma_2}), \quad \delta > (1-q)/2,$$

where $\gamma_1 = \min(3(1-q-\delta), 2(1-q+\delta))$, $\gamma_2 = \min(3(1-q-\delta), (1-q+\delta))$, and

$$\kappa^{(0)} = \frac{-\tilde{S}}{(\bar{u} - \bar{c})}, \quad \kappa^{(1)} = \frac{b_{20} - b_{11}(\bar{u} - \bar{c}) + b_{02}(\bar{u} - \bar{c})^2}{2\bar{c}^3\tilde{S}},$$

$$\kappa^{(2)} = \frac{\tilde{S}^2(\bar{\theta} + ((\gamma-1)/\gamma)\,\bar{\varphi})}{2\bar{\rho}(\bar{u} - \bar{c})^3}.$$

Note that solution (93) converges to the solution given by (89), (88), and (90) if $\delta \to 0$ and $\bar{u} \to \bar{c}$. The eigenvalues with positive real part are $\kappa_1^+ = \kappa^+$, $\kappa_2^+ = \kappa_5$, $\kappa_3^+ = \kappa_6$ and the following estimate is obtained

$$\bar{u} < \bar{c}, \quad \text{case iii.} \min_j \operatorname{Re}(\kappa_j^+) \geq C_1 \tilde{\eta} \varepsilon + C_2 \varepsilon^{2(1-q)}, \quad \min_j |\kappa_j^+| \geq C_3 \varepsilon^{1-q-\delta}. \tag{94}$$

The positive constants C_1, C_2, and C_3 are independent of ε.

Finally the case when $|S| \geq \delta_0 > 0$ as $\varepsilon \to 0$ is considered. The corrections of the undisturbed eigenvalues due to the gradient terms are small in this region of the S-plane; see (54). Lemma 2.2 with δ_0 of order one leads to

$$\bar{u} > \bar{c}, \quad \text{case iiii.} \min_j \operatorname{Re}(\kappa_j^+) \geq C_1, \quad \min_j |\kappa_j^+| \geq C_2, \tag{95}$$

where C_1 and C_2 are positive constants independent of ε.

The estimates (81), (86), (91), (94), and (95) cover the right half of the complex S-plane and the estimates (55) and (56) follow. This concludes the proof of Lemma 2.7.

Proof of Lemma 2.8. The solution is (see (28))

$$\frac{\partial^r \hat{w}}{\partial x^r} = \left(\sigma_1 - \sigma_3\left(\frac{\kappa_3}{\kappa_3 - \kappa_1}\right)\right)\psi_1 \exp\left(\frac{\kappa_1 x}{\varepsilon}\right) + \sigma_2\psi_2 \exp\left(\frac{\kappa_2 x}{\varepsilon}\right) + \sigma_3\left(\frac{\kappa_3}{\kappa_3 - \kappa_1}\right)\psi_3 \exp\left(\frac{\kappa_3 x}{\varepsilon}\right). \tag{96}$$

The same notations that were used in the proof of Lemma 2.7 will be used.

The subsonic case is analysed first. Let $S = \tilde{S}\varepsilon^{1-q+\delta}$ which corresponds to case *i* above. There is no double root in this part of the complex S-plane. The eigenvalues and eigenvectors are

$$\kappa_1^+ = \kappa_1^{(0)}\varepsilon^{1-q} + O(\varepsilon^{1-q+\delta}), \quad \kappa_2^+ = \kappa_2^{(0)} + O(\varepsilon^{1-q}),$$

$$\kappa_3^+ = \kappa_3^{(0)} + O(\varepsilon^{1-q}), \quad \delta > 0,$$

$$\psi_j(\kappa_j^+) = \psi_j^{(0)} + \psi_j^{(1)}\varepsilon^{1-q} + O(\varepsilon^{1-q+\delta}), \quad \delta > 0, j = 1, 2, 3,$$

where

$$\psi_1^{(0)} = \begin{pmatrix} \bar{\rho}\tilde{u}_y\bar{u} - \tilde{\rho}_y\{\bar{u}^2 - \bar{c}^2\} \\ -\tilde{u}_y\bar{u} \\ \{\bar{u}^2 - \bar{c}^2\}\bar{u}\kappa_1^{(0)} \\ M_\infty^2\bar{c}^2\{\{\bar{u}^2 - \bar{c}^2\}(\tilde{\rho}_y/\bar{\rho}) + (\gamma-1)\tilde{u}_y\bar{u}\} \end{pmatrix}$$

and

$$\psi_2^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_3^{(0)} = \begin{pmatrix} -\bar{\rho}/\bar{u} \\ 1 \\ 0 \\ -(\gamma-1)M_\infty^2\bar{c}^2/(\bar{u} - \bar{\varphi}\kappa_3^{(0)}/\bar{\rho}) \end{pmatrix}.$$

The determinant becomes

$$\operatorname{Det}(E) = \left\{\frac{M_\infty^2\bar{c}^2}{(\bar{u} - \bar{\varphi}\kappa_3^{(0)}/\bar{\rho})}\right\}\{A_2\} + O(\varepsilon^{1-q}),$$

where

$$A_2 = (\gamma-1)\bar{u}\tilde{u}_y\bar{\varphi}\kappa_3^{(0)}/\bar{\rho} - (\bar{u} - \bar{\varphi}\kappa_3^{(0)}/\bar{\rho})(\bar{u}^2 - \bar{c}^2)(\tilde{\rho}_y/\bar{\rho}) \tag{97}$$

By using Assumption 2.2 it can be shown that $A_2 \neq 0$ and consequently $\operatorname{Det}(E) \neq 0$.

Let $S = \tilde{S}\varepsilon^{1-q}$ which corresponds to case ii above. No double root exists in this part of the complex S-plane. The eigenvalues and eigenvectors are

$$\kappa_1^+ = \kappa_1^{(0)}\varepsilon^{1-q+O(\varepsilon^{1-q+\delta})}, \quad \kappa_2^+ = \kappa_2^{(0)} + O(\varepsilon^{1-q}),$$
$$\kappa_3^+ = \kappa_3^{(0)} + O(\varepsilon^{1-q}), \quad \delta > 0,$$
$$\psi_j(\kappa_j^+) = \psi_j^{(0)} + \psi_j^{(1)}\varepsilon^{1-q} + O(\varepsilon^{1-q+\delta}),$$
$$\delta > 0, \quad j = 1, 2, 3,$$

where

$$\psi_1^{(0)} = \begin{pmatrix} \bar{\rho}\tilde{u}_y\kappa_1^{(0)}(\bar{u}\kappa_1^{(0)} + \tilde{S}) - \tilde{\rho}_y\{(\bar{u}\kappa_1^{(0)} + \tilde{S})^2 - \bar{c}^2\kappa_1^{(0)^2}\} \\ -\tilde{u}_y(\bar{u}\kappa_1^{(0)} + \tilde{S})^2 \\ \{(\bar{u}\kappa_1^{(0)} + \tilde{S})^2 - \bar{c}^2\kappa_1^{(0)^2}\}(\bar{u}\kappa_1^{(0)} + \tilde{S}) \\ M_\infty^2\bar{c}^2\{\{(\bar{u}\kappa_1^{(0)} + \tilde{S})^2 - \bar{c}^2\kappa_1^{(0)^2}\}(\tilde{\rho}_y/\bar{\rho}) + (\gamma - 1)\tilde{u}_y\kappa_1^{(0)}(\bar{u}\kappa_1^{(0)} + \tilde{S})\} \end{pmatrix}$$

and

$$\psi_2^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_3^{(0)} = \begin{pmatrix} -\bar{\rho}/\bar{u} \\ 1 \\ 0 \\ -(\gamma - 1)M_\infty^2\bar{c}^2/(\bar{u} - \bar{\varphi}\kappa_3^{(0)}/\bar{\rho}) \end{pmatrix}.$$

The determinant becomes

$$\mathrm{Det}(E) = \left\{\frac{M_\infty^2\bar{c}^2}{(\bar{u} - \bar{\varphi}\kappa_3^{(0)}/\bar{\rho})}\right\}\{A_2\kappa_1^{(0)^2} + (A_1\tilde{S})\kappa_1^{(0)} + A_0\tilde{S}^2\} + O(\varepsilon^{1-q}),$$

where A_2 is given by Eq. (97) and

$$A_1 = (\gamma - 1)\tilde{u}_y(\bar{u} + \bar{\varphi}\kappa_3^{(0)}/\bar{\rho}) - (\bar{u} - \bar{\varphi}\kappa_3^{(0)}/\bar{\rho})2u(\tilde{\rho}_y/\bar{\rho})$$
$$A_0 = (\gamma - 1)\tilde{u}_y - (\tilde{u} - \bar{\varphi}\kappa_3^{(0)}/\bar{\rho})(\tilde{\rho}_y/\bar{\rho}).$$

An eigenvalue of the form $\kappa_1^{(0)} = C_1\tilde{S}$ solves $A_2\kappa_1^{(0)^2} + (A_1\tilde{S})\kappa_1^{(0)} + A_0\tilde{S}^2 = 0$. If $\mathrm{Re}(C_1) > 0$ a singular matrix E might exist. The introduction of $\kappa_1^{(0)} = C_1\tilde{S}$ into (87) leads to

$$\{a_{40}C_1^4 + a_{31}C_1^3 + a_{22}C_1^2 + a_{13}C_1 + a_{04}\}\tilde{S}^4 + \{b_{20}C_1^2 + b_{11}C_1 + b_{02}\}\tilde{S}^2 = 0.$$

If the coefficients multiplying $\tilde{S}^4$ and $\tilde{S}^2$ both vanish for $\mathrm{Re}(C_1) > 0$ we might have $\mathrm{Det}(E) = O(\varepsilon^{1-q})$. However (see (83), (84), (85)) the solutions to $b_{20}Z^2 + b_{11}Z + b_{02} = 0$ lead to $\mathrm{Re}(Z) < 0$ and, consequently, Det $(E) \neq 0$ for case ii in subsonic flow.

In case iii, i.e., $S = \tilde{S}\varepsilon^{1-q-\delta}$, the following eigenvalues and principal part of the eigenvectors are obtained:

$$\kappa_1^+ = \kappa_1^{(0)}\varepsilon^{1-q-\delta} + O(\varepsilon^{1-q}), \quad 0 < \delta < 1 - q,$$
$$\kappa_2^+ = \kappa_2^{(0)} + O(\varepsilon^{1-q-\delta})$$
$$\kappa_3^+ = \kappa_3^{(0)} + O(\varepsilon^{1-q-\delta})$$

$$\psi_1^{(0)} = \begin{pmatrix} \bar{\rho} \\ -\bar{c} \\ 0 \\ (\gamma - 1)M_\infty^2\bar{c}^2 \end{pmatrix}, \quad \psi_2^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\psi_3^{(0)} = \begin{pmatrix} -\bar{\rho}/\bar{u} \\ 1 \\ 0 \\ -(\gamma - 1)M_\infty^2\bar{c}^2/(\bar{u} - \bar{\varphi}\kappa_3^{(0)}/\bar{\rho}) \end{pmatrix}.$$

The determinant becomes

$$\mathrm{Det}(E) = \left\{\frac{(\gamma - 1)M_\infty^2\bar{c}^2}{(\bar{u} - \bar{\varphi}\kappa_3^{(0)}/\bar{\rho})}\right\}\{\bar{c} - \bar{u} + \bar{\varphi}\kappa_3^{(0)}/\bar{\rho}\} + O(\varepsilon^{1-q}).$$

It follows that $\mathrm{Det}(E) \neq 0$ since $\kappa_3^{(0)} > 0$.

Let us now turn to the supersonic case. Cases i–iii can be treated simultaneously. The eigenvalues and principal part of the eigenvectors are

$$\kappa_1^+ = \kappa_1^{(0)} + O(\varepsilon^\delta), \quad \kappa_2^+ = \kappa_2^{(0)} + O(\varepsilon^\delta),$$
$$\kappa_3^+ = \kappa_3^{(0)} + O(\varepsilon^\delta), \quad \delta > 0,$$

$$\psi_j^{(0)} = \begin{pmatrix} -\bar{\rho}/\bar{u} \\ 1 \\ 0 \\ -(\gamma - 1)M_\infty^2\bar{c}^2/(\bar{u} - \bar{\varphi}\kappa_j^{(0)}/\bar{\rho}) \end{pmatrix}, \quad j = 1, 3 \quad \psi_2^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The determinant becomes

$$\mathrm{Det}(E) = \left\{\frac{-(\gamma - 1)M_\infty^2\bar{c}^2(\bar{\varphi}\kappa_3^{(0)}/\bar{\rho})}{(\bar{u} - \bar{\varphi}\kappa_3^{(0)}/\bar{\rho})(\bar{u} - \bar{\varphi}\kappa_1^{(0)}/\bar{\rho})}\right\} + O(\varepsilon^\delta)$$

and the determinant is obviously non-zero.

Finally the case when $|S| \geq \delta_0 > 0$ as $\varepsilon \to 0$ is considered. The corrections of the undisturbed eigenvalues due to the gradient terms are small in this region of the S-plane (see (54)). Let $\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\kappa}_3$ be the roots of the undisturbed problem. The leading terms of the eigenvectors in the case of distinct roots $\tilde{\kappa}_1 \neq \tilde{\kappa}_3$ are

$$\psi_1 = r(\tilde{\kappa}_1) + O(\varepsilon^{(1-q)}), \quad \psi_2 = e_3 + O(\varepsilon^{(1-q)}), \quad \psi_3 = r(\tilde{\kappa}_3) + O(\varepsilon^{(1-q)}), \tag{98}$$

where $r(\kappa)$ and e_3 are given by (26) and (27). A direct calculation using the formulation (96) for distinct roots leads to

$$\mathrm{Det}(E) = \mathrm{Det}(E_s) + O(\varepsilon^{1-q}), \quad \lim_{\varepsilon\to 0} \mathrm{Det}(E) = \mathrm{Det}(E_s), \tag{99}$$

where $\mathrm{Det}(E_s)$ is the determinant for distinct roots in the undisturbed case given by (36). Furthermore (see (39)), for $|\tilde{\kappa}_3 - \tilde{\kappa}_1| \to 0$ we have $\mathrm{Det}(E_s) = \mathrm{Det}(E_d)$. The determinant for double roots in the undisturbed case, $\mathrm{Det}(E_d)$, is given by (38). By using (99), (39), and Lemma 2.5, it follows that condition (58) holds. This concludes the proof of Lemma 2.8.

Proof of Lemma 2.9. Let us first consider the case when $|S| \to 0$ as $\varepsilon \to 0$. No double roots exist in this part of the complex S-plane. The solution is

$$\frac{\partial^r \hat{w}}{\partial x^r} = \mathrm{R}_1 \psi_1 \exp\left(\frac{\kappa_1 x}{\varepsilon}\right) + \mathrm{R}_2 \psi_2 \exp\left(\frac{\kappa_2 x}{\varepsilon}\right) + \mathrm{R}_3 \psi_3 \exp\left(\frac{\kappa_3 x}{\varepsilon}\right),$$

where

$$\mathrm{R}_1 = -\sigma_3\left(\frac{\kappa_3}{\kappa_3 - \kappa_1}\right) + \sigma_1, \quad \mathrm{R}_2 = \sigma_2, \quad \mathrm{R}_3 = +\sigma_3\left(\frac{\kappa_3}{\kappa_3 - \kappa_1}\right).$$

Condition (58) and the fact that the eigenvectors ψ_1, ψ_2, and ψ_3 are bounded in this part of the complex S-plane leads to

$$|\mathrm{R}_1| \le \mathrm{const}\, |\hat{g}|, \quad |\mathrm{R}_2| \le \mathrm{const}\, |\hat{g}|, \quad |\mathrm{R}_3| \le \mathrm{const}\, |\hat{g}|.$$

Integration of $\partial^r \hat{w}/\partial x^r$ from $-\infty$ to x leads to

$$\hat{w} = \mathrm{R}_1\left(\frac{\varepsilon}{\kappa_1}\right)^r \psi_1 \exp\left(\frac{\varepsilon}{\kappa_1}\right) + \mathrm{R}_2\left(\frac{\varepsilon}{\kappa_2}\right)^r \psi_2 \exp\left(\frac{\varepsilon}{\kappa_2}\right) + \mathrm{R}_3\left(\frac{\varepsilon}{\kappa_3}\right)^r \psi_3 \exp\left(\frac{\varepsilon}{\kappa_3}\right)$$

and condition (59) follows.

Next, the case where $|S| \ge \delta_0 > 0$ as $\varepsilon \to 0$ is considered. Let $\tilde{\kappa}_1$, $\tilde{\kappa}_2$, $\tilde{\kappa}_3$ be the roots of the undisturbed problem. The corrections of the undisturbed eigenvalues due to the gradient terms are small in this region of the S-plane (see (54)). The leading terms of the eigenvectors in the case of distinct roots $\tilde{\kappa}_1 \neq \tilde{\kappa}_3$ are given by (98). A direct calculation using the formulation (96) for distinct roots leads to

$$\begin{aligned} \partial^r \hat{w}/\partial x^r = {} & \{H_1 + O(\varepsilon^{1-q})\} \exp\left\{\left(\frac{\tilde{\kappa}_1 x}{\varepsilon}\right)(1 + O(\varepsilon^{2(1-q)}))\right\} \\ & + \{H_2 + O(\varepsilon^{1-q})\} \exp\left\{\left(\frac{\tilde{\kappa}_2 x}{\varepsilon}\right)(1 + O(\varepsilon^{2(1-q)}))\right\} \\ & + \{H_3 + O(\varepsilon^{1-q})\} \exp\left\{\left(\frac{\tilde{\kappa}_3 x}{\varepsilon}\right)(1 + O(\varepsilon^{2(1-q)}))\right\}, \end{aligned}$$

where H_1, H_2, H_3 are given by (44). Integration of $\partial^r \hat{w}/\partial x^r$ from $-\infty$ to x leads to

$$\begin{aligned} \hat{w} = {} & \left\{\left(\frac{\tilde{\kappa}_1 x}{\varepsilon}\right)(1 + O(\varepsilon^{2(1-q)}))\right\}^{-r} \{H_1 + O(\varepsilon^{1-q})\} \\ & \exp\left\{\left(\frac{\tilde{\kappa}_1 x}{\varepsilon}\right)(1 + O(\varepsilon^{2(1-q)}))\right\} \\ & + \left\{\left(\frac{\tilde{\kappa}_2 x}{\varepsilon}\right)(1 + O(\varepsilon^{2(1-q)}))\right\}^{-r} \{H_2 + O(\varepsilon^{1-q})\} \\ & \exp\left\{\left(\frac{\tilde{\kappa}_2 x}{\varepsilon}\right)(1 + O(\varepsilon^{2(1-q)}))\right\} \\ & + \left\{\left(\frac{\tilde{\kappa}_3 x}{\varepsilon}\right)(1 + O(\varepsilon^{2(1-q)}))\right\}^{-r} \{H_3 + O(\varepsilon^{1-q})\} \\ & \exp\left\{\left(\frac{\tilde{\kappa}_1 x}{\varepsilon}\right)(1 + O(\varepsilon^{2(1-q)}))\right\}. \end{aligned} \tag{100}$$

Lemma 2.6 states that the solution of the undisturbed problem satisfies the estimate (59). The solution given by (100) converges to the undisturbed solution for distinct roots given by (43) as $\varepsilon \to 0$. Moreover, Lemma 2.4 states that the undisturbed solution for distinct roots with $\tilde{\kappa}_1 \neq \tilde{\kappa}_3$ converges to the undisturbed solution for a double root $\tilde{\kappa}_1 = \tilde{\kappa}_3 = \tilde{\kappa}$ as $\tilde{\kappa}_3 \to \tilde{\kappa}_1$ and, hence, we can conclude also that the solution of the disturbed problem satisfies the estimate (59). This concludes the proof of Lemma 2.9.

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